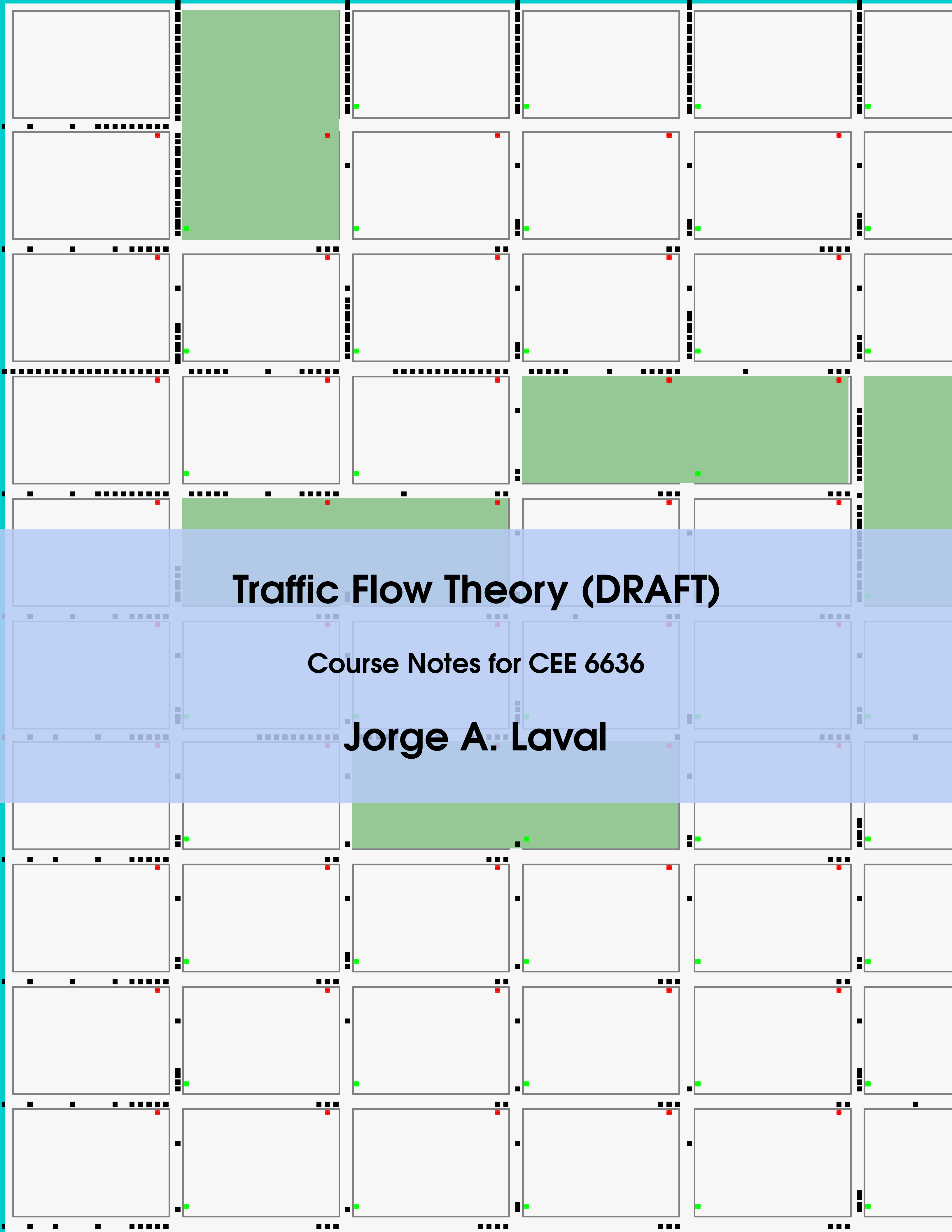

Traffic Flow Theory (DRAFT)

Course Notes for CEE 6636

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This course presents a comprehensive overview on vehicular traffic flow theory and its use in evaluating congestion and determining control strategies. Starting from the basic concepts defining traffic streams, existing theories are presented, including the kinematic wave model, cellular automata models, and car-following (microsimulation) models.

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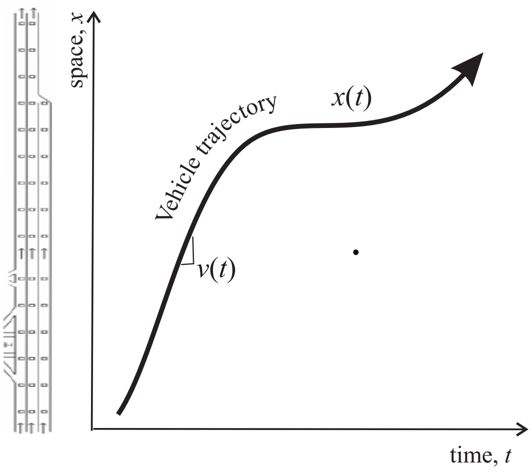
Part One

1. Basic concepts

1.1 Time-space diagram

PowerPoint presentation (intro)

PowerPoint presentation (t-x)



1.2 Single vehicle dynamics

1.2.1 Motion as a function of time

Let $x(t)$ be the vehicle trajectory, i.e. its position at time t. Then,

$$v(t) = x'(t) \qquad \text{(speed)} \qquad (1.1a)$$

$$a(t) = v'(t) = x''(t) \qquad \text{(acceleration)} \qquad (1.1b)$$

$$j(t) = a'(t) = v''(t) = x'''(t) \qquad \text{(jerk)} \qquad (1.1c)$$

Or, equivalently,

$$x(t) = x_0 + \int_{t_0}^{t} v(s)ds \tag{1.2a}$$

$$v(t) = v_0 + \int_{t_0}^{t} a(s)ds \tag{1.2b}$$

$$a(t) = a_0 + \int_{t_0}^{t} j(s)ds \tag{1.2c}$$

where all the variables with the subscript "0" are given initial conditions at time t_0. Note e.g. that:

$$x(t) = x_0 + \int_{t_0}^{t} (v_0 + \int_{t_0}^{s} a(r)dr)ds = x_0 + v_0(t - t_0) + \int_{t_0}^{t} \int_{t_0}^{s} a(r)dr\, ds \tag{1.3}$$

Example 1.1. For constant speed, $v(x) = v$, Eq. (1.2a) gives

$$x(t) = x_0 + (t - t_0)v$$

■

Example 1.2. For constant acceleration $a(t) = b$, Eq. (1.3) gives

$$\begin{aligned} v(t) &= v_0 + (t - t_0)b \\ x(t) &= x_0 + v_0(t - t_0) + \frac{1}{2}b(t - t_0)^2 \end{aligned}$$

With initial conditions (x_0, v_0, t_0)=0, the expression simplifies to $x(t) = \frac{1}{2}bt^2$. ■

Example 1.3. — Braking distance Suppose $a(t) = -(bt + c), b, c > 0$. Calculate the braking distance of a vehicle traveling at $v_0 = 20$m/s if the reaction time is $t_0 = 1$s and $b = 1.7$ m/s^3, $c = 7$ m/s^2. (ans=40.3599 m, solved in class)
→ Click for solution in Mathematica. ■

1.2.2 Motion as a function of distance

In some applications it is convenient to take distance as the independent variable. A vehicle trajectory is represented by $t(x)$, the *inverse function* of $x(t)$.

If $v(x)$ is given, $t(x)$ can be derived noting that $v(x) = dx/dt$ and we have

$$dt = \frac{dx}{v(x)}, \qquad \text{which by integration gives}$$

$$\boxed{t(x) = t_0 + \int_{x_0}^{x} \frac{dx}{v(x)}.} \tag{1.4}$$

Example 1.4. For constant speed, $v(x) = v$, Eq. (1.4) gives

$$t(x) = t_0 + \frac{x - x_0}{v}$$

This result also corresponds to the inverse function of $x(t)$ in Example 1.1. ■

If $a(x)$ is given, $v(x)$ can be derived using the chain rule: $a(x) = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$, or:

$$a(x) = v'(x)v(x) \tag{1.5}$$

This can also be written as $a(x) = \frac{d[\frac{1}{2}v^2]}{dx}$, or better as $d[\frac{1}{2}v(x)^2] = a(x)dx$, which can be integrated to give $v(x)^2 = v_0^2 + 2\int_{x_0}^{x} a(x)dx$. Hence,

$$\boxed{v(x) = \sqrt{v_0^2 + 2\int_{x_0}^{x} a(x)dx.}} \tag{1.6}$$

Notice that (1.4) and (1.6) are the solution of the following system:

$$\begin{cases} v'(x) = a(x)/v(x), \qquad v(x_0) = v_0 & (1.7a) \\ t'(x) = 1/v(x), \qquad t(x_0) = t_0 & (1.7b) \end{cases}$$

which can be used for if a computer program is used to obtain the solution.

Example 1.5. Consider a motion with $a(t) = a(x) = b =$const. According to Eq. (1.6)

$$v(x) = \sqrt{v_0^2 + 2b(x - x_0)}$$

Then $t(x)$ can be calculated in two ways:
1. Using $v(x)$ and Eq. (1.4) we have

$$\begin{aligned} t(x) &= t_0 + \int_{x_0}^{x} \frac{dx}{v(x)} \\ &= t_0 + \int_{x_0}^{x} \frac{dx}{\sqrt{v_0^2 + 2b(x - x_0)}}. \end{aligned}$$

It follows that

$$\begin{aligned} t(x) &= t_0 + \frac{1}{2b \times \frac{1}{2}} \sqrt{v_0^2 + 2b(x - x_0)}\Big|_{x_0}^{x} \\ &= t_0 - \frac{v_0}{b} + \frac{1}{b}\sqrt{v_0^2 + 2b(x - x_0)}. \end{aligned}$$

2. Inverting the function $x(t)$ in Example 1.2:

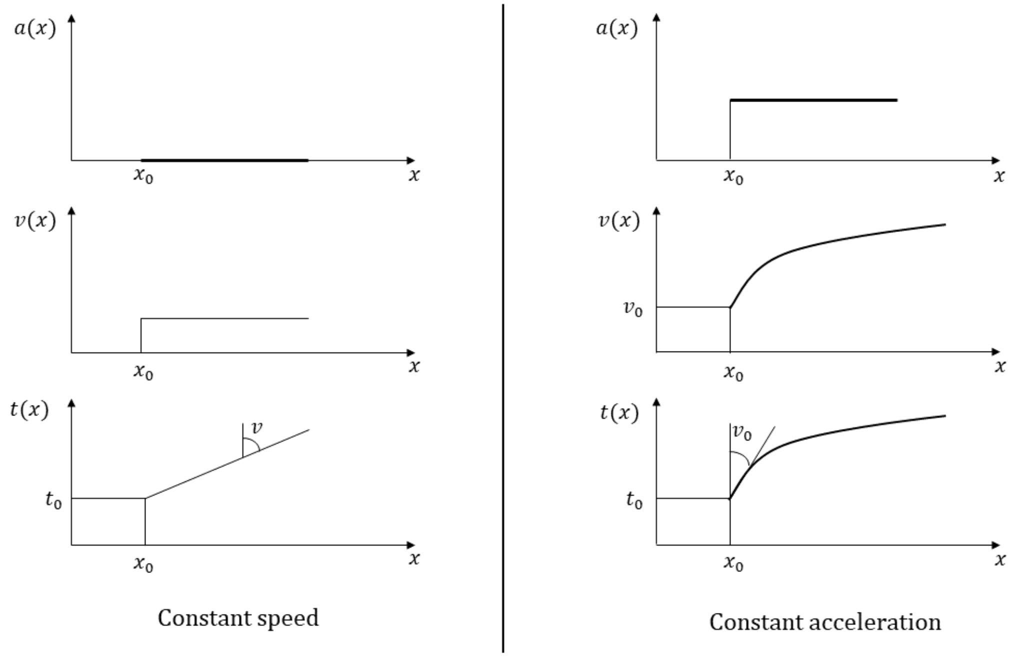
$$x(t) - x_0 = \frac{1}{2}b(t - t_0)^2 + v_0(t - t_0)$$

gives the function $t(x)$, above.

In any case, with initial conditions (t_0, v_0)=0, the expression simplifies to

$$t(x) = \frac{1}{b}\sqrt{2b(x-x_0)}.$$

■



Example 1.6. Consider a vehicle moving with a distance-dependent acceleration given by:

$$a(x) = x$$

Show that:

$$\begin{aligned} v(x) &= \sqrt{v_0^2 + x^2} \\ t(x) &= \log\left(\frac{\sqrt{v_0^2+x^2}+x}{v_0}\right). \end{aligned}$$

■

1.2.3 Motion as a function of speed: Vehicle kinematics models

Vehicle kinematics models give the “desired acceleration":

$$a = a(v), \tag{1.8}$$

that the driver imposes to the vehicle when traveling at a speed $v(t)$ at time t under free-flow conditions, i.e. when unobstructed by the leading vehicle. A desired acceleration model captures both driver behavior and the physical limitations imposed by roadway geometry on the engine.

Noting that $a(v) = dv/dt$ we have

$$dt = \frac{dv}{a(v)}, \qquad \text{which by integration gives}$$

$$\boxed{t(v) = t_0 + \int_{v_0}^{v} \frac{dv}{a(v)}.} \tag{1.9}$$

The time-dependent description is obtained by inverting (1.9) and using the equations of motion.

The position $x(v)$ can be derived using the chain rule:

$$a(v) = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v \tag{1.10}$$

This gives $dx = \frac{v}{a(v)}dv$ and hence

$$\boxed{x(v) = x_0 + \int_{v_0}^{v} \frac{v}{a(v)}dv} \tag{1.11}$$

Notice that (1.9) and (1.11) are the solution of the following system:

$$\begin{cases} t'(v) = 1/a(v), & t(v_0) = t_0 \quad (1.12a) \\ x'(v) = v/a(v), & x(v_0) = x_0 \quad (1.12b) \end{cases}$$

The distance-dependent description:

$$\begin{cases} v'(x) = a(v(x))/v(x), & v(x_0) = v_0 \quad (1.13a) \\ t'(x) = 1/v(x), & t(x_0) = t_0 \quad (1.13b) \end{cases}$$

Example 1.7. For constant acceleration, $a(v) = b$, Eq. (1.9) gives

$$t(v) = t_0 + \frac{v - v_0}{b}$$

which corresponds to the inverse function of $v(t)$ in Example 1.2.

Eq. (1.11) gives

$$\begin{aligned} x(v) &= x_0 + b\int_{v_0}^{v} v dv \\ &= x_0 + \frac{v^2 - v_0^2}{2b} \end{aligned}$$

which corresponds to the inverse function of $v(x)$ in Example 1.5. ■

Example 1.8. Consider a vehicle moving with a constant-power acceleration given by:

$$a(v) = 1/v$$

Show that:

$$
\begin{aligned}
t(v) &= \frac{1}{2}\left(2t_0 + v^2 - v_0^2\right) \\
x(v) &= \frac{1}{3}\left(v^3 - v_0^3\right) + x_0.
\end{aligned}
$$

How can we obtain a time-dependent description? To obtain the time-dependent description, one can take the inverse of (1.9) to obtain $v(t)$ and proceed as usual using (1.1) and (1.2) to obtain $x(t)$ and $a(t)$. Alternatively, since $a(t) = v'(t)$ eqn. (1.8) becomes

$$v'(t) = a(v(t)), \tag{1.14}$$

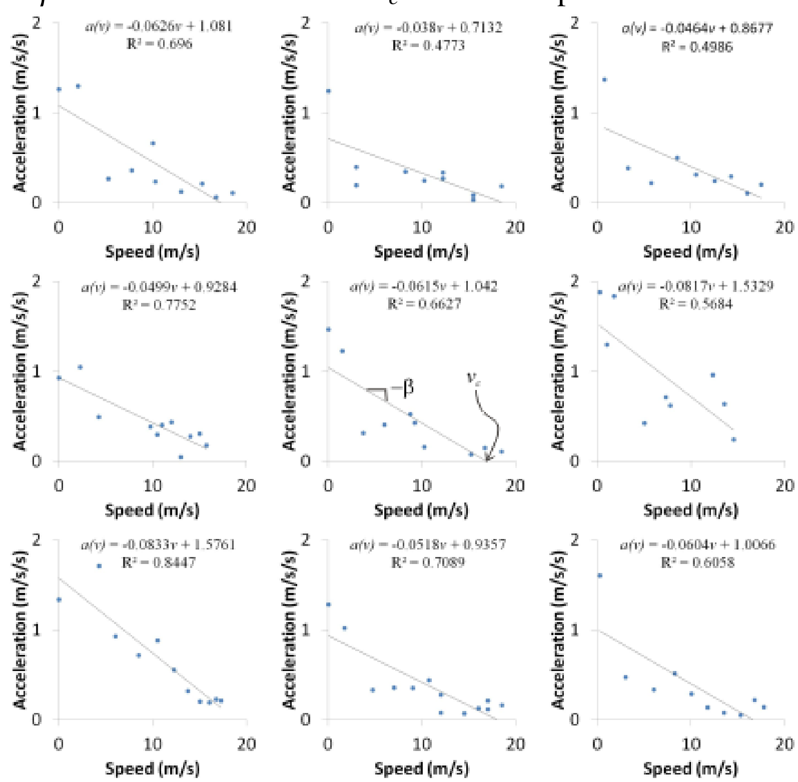
which is a first-order ordinary differential equation (ODE) in $v(t)$ with initial condition $v(t_0) = v_0$, which can be solved *numerically* with standard ODE methods.

1.2.4 Linear acceleration model

For light vehicles, a good approximation is a linearly decreasing function of the speed:

$$\boxed{\frac{dv}{dt} = (v_c - v)\beta} \qquad \text{(linear acc. model)} \tag{1.15}$$

where β has units of time^{-1} and v_c can be interpreted as a desired speed.



This figure shows a sample of the data collection effort in this paper that justifies a linear desired acceleration model. The data was collected during the acceleration process at signalized intersections where the vehicle was the leader of the platoon. This figure also suggests that $v_c \approx 18$m/s, $\beta \approx 0.06$s^{-1} and $a_m \approx 1$m/s^2 are reasonable typical values.

Time-dependent solution. The ODE (1.15) can be solved analytically using (1.9) to get:

$$t(v) = \frac{1}{\beta} \log\left(\frac{v_c - v_0}{v_c - v}\right)$$

and taking the inverse:

$$\boxed{v(t) = v_c - e^{-\beta t}(v_c - v_0),} \tag{1.16}$$

where we used $t_0 = 0$. We then use (1.1) and (1.2) to obtain

$$x(t) = x_0 + t v_c - (v_c - v_0)(1 - e^{-\beta t})/\beta, \tag{1.17a}$$
$$a(t) = \beta e^{-\beta t}(v_c - v_0), \tag{1.17b}$$
$$j(t) = -\beta^2 e^{-\beta t}(v_c - v_0). \tag{1.17c}$$

Distance-dependent solution.

$$v(x) = v_c(W(B(x)) + 1), \tag{1.18a}$$
$$t(x) = \frac{A(x) + W(B(x))}{\beta}, \tag{1.18b}$$
$$a(x) = -\beta v_c W(B(x)). \tag{1.18c}$$

where $W(\cdot)$ is the Lambert W-Function (aka Product-Log function) and

$$A(x) = \frac{v_c - v_0 + \beta x}{v_c}, \qquad B(x) = \frac{(v_0 - v_c)\, e^{-A(x)}}{v_c}.$$

Mathematica code:

```
A[x_] := -(v0 - vc - b x)/vc
B[x_] := (E^(-A[x]) (v0 - vc))/vc
t[x_] := (A[x] + ProductLog[B[x]])/(b)
v[x_] := vc (1 + ProductLog[B[x]])
a[x_] := -b vc ProductLog[B[x]]
```

At this point, one could say that the problem is solved because equations (1.16) and (1.17) gives us everything we need to know about the motion of the accelerating driver. But to really understand the model, one should know how each parameter, β, v_c and v_0 in this case, affect the solution, which may not be an easy task.

Dimensionless formulation (Optional)

Dimensionless formulations are convenient because they reduce the number of parameters involved in a problem. In the present case of the linear desired acceleration model (1.15), define

$$\tilde{t} = \beta t, \quad \text{and} \quad \tilde{v} = v/v_c, \tag{1.19}$$

which means that we measure time in units of β^{-1} (instead of, say, seconds), speed in units of v_c (instead of km/hr). The quantity

$$\tau = \beta^{-1}$$

is the time scale of the problem, a.k.a. relaxation time, characteristic time. This means that the time for the system to reach equilibrium from a perturbation is comparable to τ.

The corresponding transformation for the space variable

$$\widetilde{x} = \int_{\widetilde{t}_0}^{\widetilde{t}} \widetilde{v}(\widetilde{s}) d\widetilde{s} = \frac{1}{\tau v_c} \int_{t_0}^{t} v(s) ds = x(t)/(\tau v_c)$$

is obtained by a change of variable noting that $d\tilde{s} = \beta ds$.

The ODE (1.15) is now

$$\boxed{\widetilde{v}' = 1 - \widetilde{v}} \tag{1.20}$$

with initial condition $\widetilde{v}(0) = \widetilde{v}_0$. Setting $x_0 = 0$ the equations of motion become

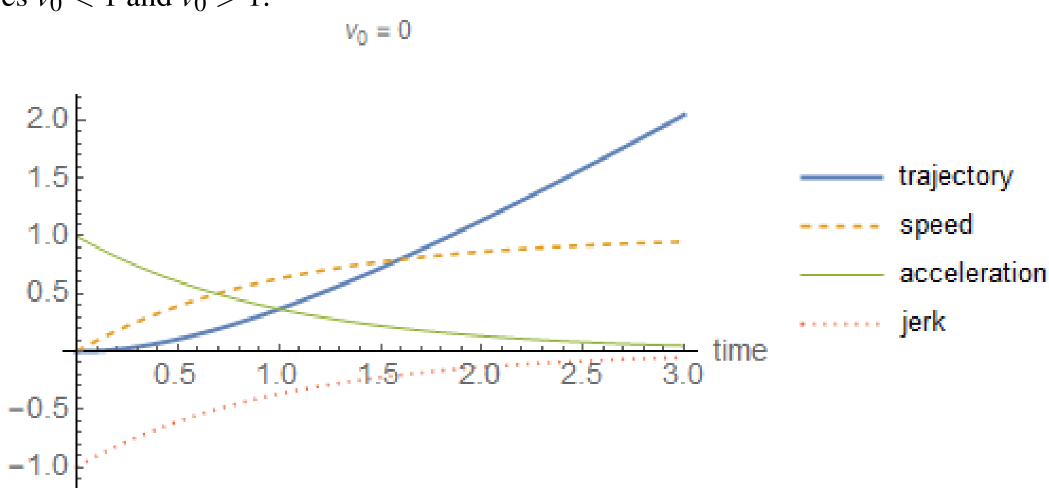
$$\widetilde{x}(\widetilde{t}) = \widetilde{t} - (1 - \widetilde{v}_0)(1 - e^{-\widetilde{t}}), \tag{1.21a}$$

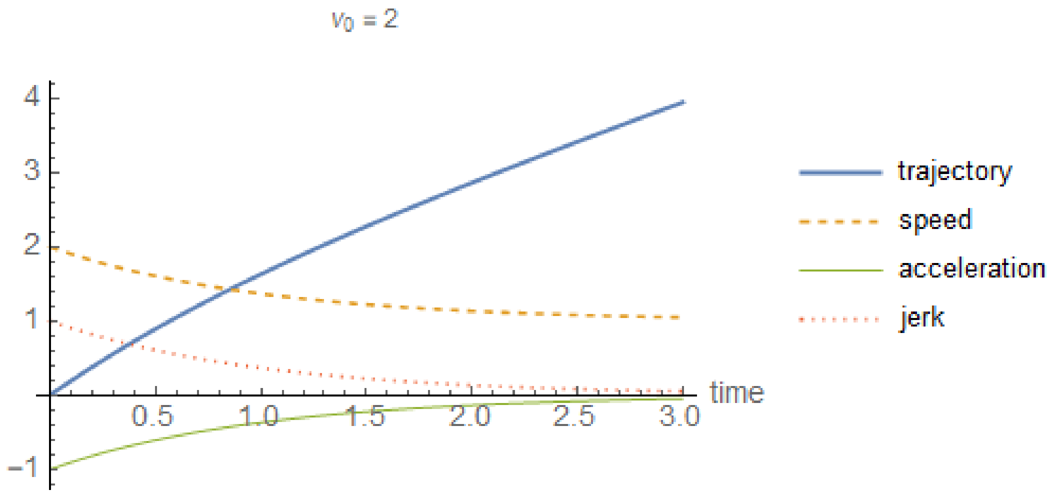
$$\widetilde{v}(\widetilde{t}) = 1 - e^{-\widetilde{t}}(1 - \widetilde{v}_0), \tag{1.21b}$$

$$\widetilde{a}(\widetilde{t}) = e^{-\widetilde{t}}(1 - \widetilde{v}_0), \tag{1.21c}$$

$$\widetilde{j}(\widetilde{t}) = -e^{-\widetilde{t}}(1 - \widetilde{v}_0), \tag{1.21d}$$

and the only parameter is the initial condition $\widetilde{v}_0$. The following two figures illustrate the two possible cases $\widetilde{v}_0 < 1$ and $\widetilde{v}_0 > 1$.





A completely parameter-free formulation is given by the transformation

$$\tilde{t} = \beta t, \quad \text{and} \quad \tilde{v} = \frac{v_c - v}{v_c - v_0}, \tag{1.22}$$

The ODE becomes $\widetilde{v}' = -\widetilde{v}$ with initial condition $\widetilde{v}(0) = 1$; this gives

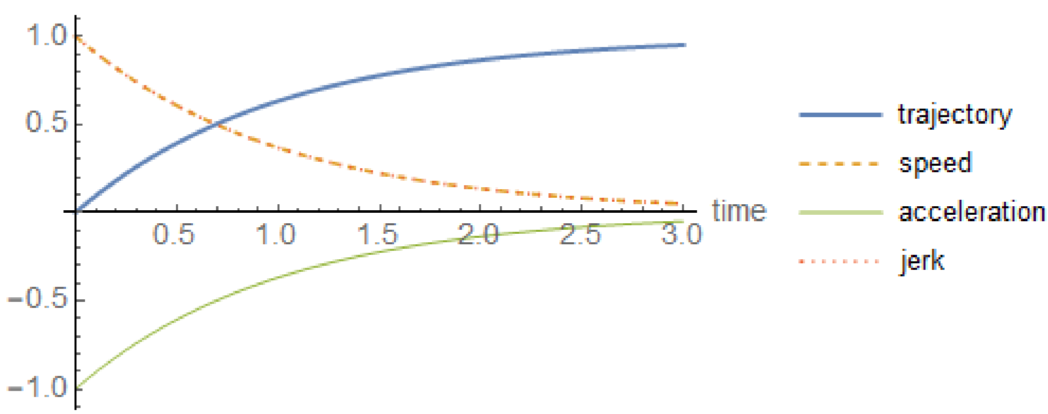
$$\widetilde{x}(\widetilde{t}) = 1 - e^{-\widetilde{t}}, \tag{1.23a}$$

$$\widetilde{v}(\widetilde{t}) = e^{-\widetilde{t}}, \tag{1.23b}$$

$$\widetilde{a}(\widetilde{t}) = -e^{-\widetilde{t}} \tag{1.23c}$$

$$\widetilde{j}(\widetilde{t}) = e^{-\widetilde{t}}, \tag{1.23d}$$

These results need to be interpreted carefully, but the advantage is that the model can be summarized in a single figure, shown below, that describes the “universal shape" of linear acceleration models.



1.2.5 Vertical and horizontal curves

Real-life roadways tend to be very different from the idealized straight and flat case we’ve seen so far. Vertical and horizontal curves have been at different locations along the road, and influence the driving behavior such that the acceleration is now a function of both the speed and the location. Therefore, the equations of motion are now given by the following system of ODEs:

$$\begin{cases} v'(t) = a(v(t), x(t)), & v(t_0) = v_0 \quad (1.24a) \\ x'(t) = v(t), & x(t_0) = x_0 \quad (1.24b) \end{cases}$$

A distance-dependent description should be easier to solve than (1.24). Using (1.7) gives the following system:

$$\begin{cases} v'(x) = a(v(x),x)/v(x), & v(x_0) = v_0 \\ t'(x) = 1/v(x), & t(x_0) = t_0 \end{cases} \quad \begin{matrix} (1.25a) \\ (1.25b) \end{matrix}$$

Notice that the ODEs in this system are independent of each other, as opposed to (1.24). Therefore, one could try to solve (1.25a) analytically and then obtain $t(x)$ using (1.4) and $a(x)$ using (1.5).

Example 1.9. Consider a vehicle moving with a distance-dependent acceleration given by:

$$a(v,x) = x/v$$

Show that if $t_0 = x_0 = v_0 = 0$ then:

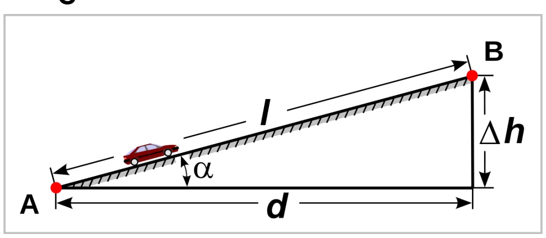
$$\begin{aligned} v(x) &= \sqrt[3]{\frac{3x^2}{2}}, \qquad t(x) = \sqrt[3]{18x} \\ a(x) &= \sqrt[3]{\frac{2x}{3}}, \qquad x(t) = t^3/18 \end{aligned}$$

Mathematica code:

```
a[v_, x_] := x/v
system = { v'[x] == a[v[x], x]/v[x], v[0] == 0};
sol = DSolve[system, v, x]
```

■

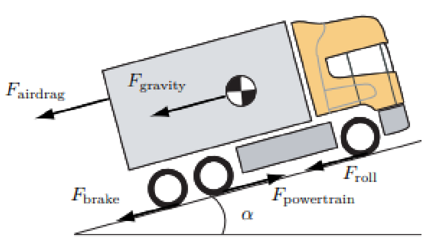
Uphill grades



→ Wikipedia

d = run, Δh = rise, l = slope length, α = angle of inclination

$G = \tan\alpha = \Delta h/d$



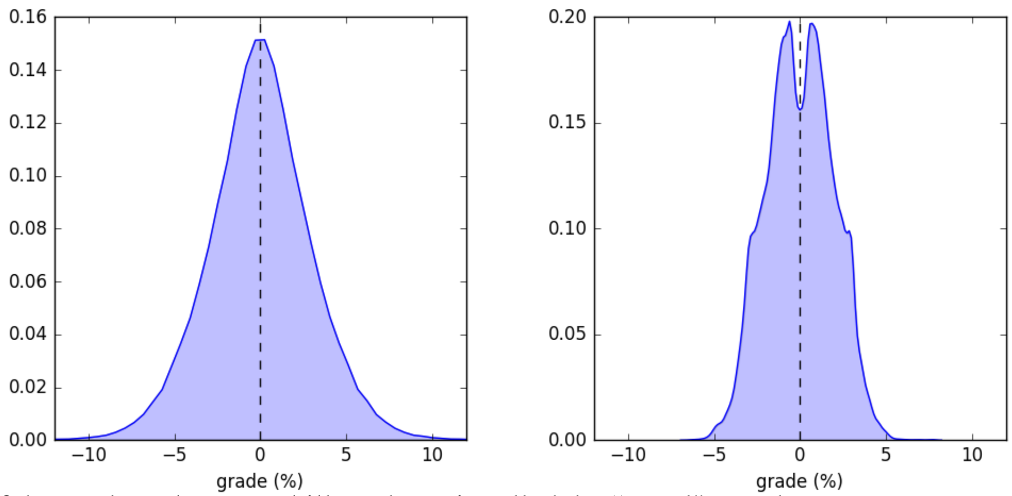
The acceleration due to gravity in the opposite direction of motion is:

$$g\sin(\alpha) \approx g\alpha \approx gG \tag{1.26}$$

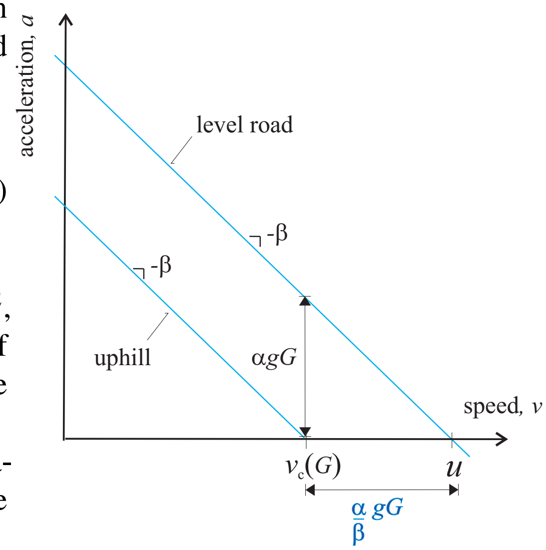
because α is small for typical roads.

Grade histogram in Atlanta:
Arterials

Freeways



If the roadway has an uphill grade, v_c is called the "crawl" speed. It is the steady state speed of a vehicle on an infinite upgrade, where the speed drops to a point where the engine lacks power to accelerate. For a particular vehicle type, it is solely a function of the grade. For the linear acceleration model it can be assumed to be given by



$$v_c(x) = u - \frac{\alpha}{\beta} gG(x) \qquad \text{(crawl speed)} \qquad (1.27)$$

where u is the desired speed on a level roadway, $g = 9.81$ m/s^2, $G(x)$ is the grade at x (as a decimal), and α is the proportion of the gravitational forces due to the uphill that are "felt" by the trajectory:

- $\alpha = 0$: drivers compensate completely for the gravitational forces (by pressing the gas pedal harder) so that the uphill does not influence the trajectory,
- $\alpha = 1$: drivers do not change their behavior with respect to a level road in the full effect of gravitational forces will slow down the vehicle
- Preliminary data: $\alpha \approx 0.6$ for cars and $\alpha \approx 0.95$ for trucks.

R To get an idea, take as a rough approximation $\beta \approx 0.05\text{s}^{-1}$ and $G = 0.05$ to see that on a steep freeway uphill, cars lose about 10 m/s or 1/3 of their speed.

The system (1.24) becomes:

$$\begin{cases} v'(t) = (v_c(x(t)) - v(t))\beta, & v(t_0) = v_0 \qquad (1.28a) \\ x'(t) = v(t), & x(t_0) = x_0 \qquad (1.28b) \end{cases}$$

and the distance-dependent description is:

$$\begin{cases} v'(x) = (v_c(x) - v(x))\beta / v(x), & v(x_0) = v_0 \qquad (1.29a) \\ t'(x) = 1/v(x), & t(x_0) = t_0 \qquad (1.29b) \end{cases}$$

Downgrades

→ paper

Horizontal curves

The radius R for a horizontal curve is related to the design speed V, the coefficient of friction f_s, and the allowed superelevation on the curve, e:

$$R = \frac{V^2}{g\,(e + f_s)}$$

.... → Wikipedia link

1.2.6 Application: Eco-driving in gas-powered vehicles

Here we approximate the energy of a single acceleration process and derive an eco-driving strategy that minimizes emissions. Instantaneous vehicle specific power (VSP) is the main input needed for energy and emissions calculations. In MOVES, the VSP is calculated as:

$$\boxed{VSP(a,v) = v\left(a + g\,G + \frac{A}{M} + \frac{B}{M}v + \frac{C}{M}v^2\right)} \tag{1.30}$$

where:

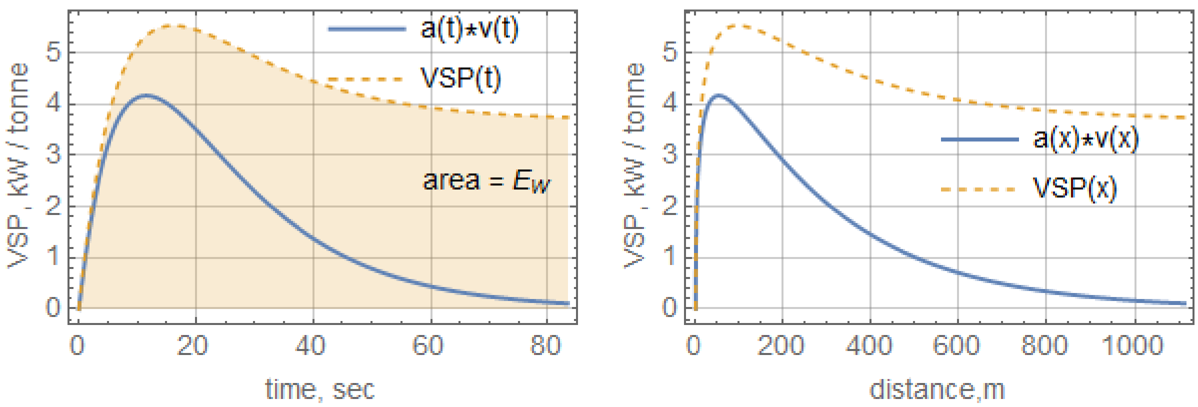
VSP = vehicle specific power a.k.a. power to weight ratio in kW/tonne (*at the wheels*)
v = $v(t)$ or $v(x)$ in m/sec
a = $a(t)$ or $a(x)$ in m/sec^2
g = 9.81 m/sec^2
G = road grade, as a decimal
A = rolling resistance (kW-sec/m)
B = rotating resistance (kW-sec$^2/m^2$)
C = aerodynamic drag (kW-sec$^3/m^3$)
M = vehicle mass (tonnes)

The value of parameters A, B, C and M are tabulated for 13 vehicle types in MOVES. For a passenger car: A = 0.156461; B = 0.00200193; C = 0.000492646; M = 1.4788. The MOVES model uses a binning approach in emissions modeling, each bin corresponding to a *discrete* range of speed, VSP and whether the vehicle is braking, idle, or in cruise-acceleration. Once the bin is known, there are lookup tables to obtain the corresponding energy and emissions. In this example we simplify this binning approach using linear regression approximations of the MOVES tables.

We now focus on a single acceleration process of a typical passenger car using the linear acceleration model (1.15), starting from rest $v_0 = 0$ to the desired speed $v_c = 60$km/hr on a flat road. We use $\beta = 0.06\text{s}^{-1}$. We know that the analytical solution for the equations of motions of the vehicle are given by (1.16) and (1.17) as a function of time, and by (1.18) is a function of distance.

Power

The power consumed is shown in the next two figures as a function of time and distance, respectively.



Energy

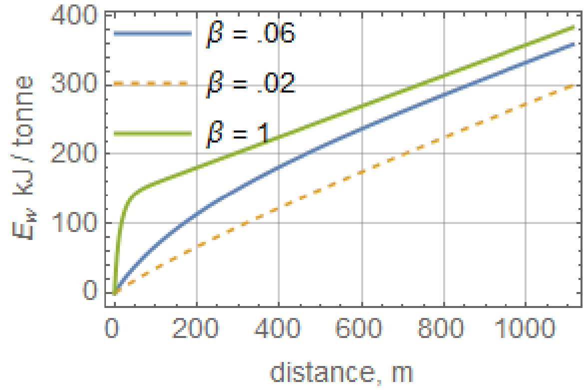
The energy is the time integral of power. Since VSP is defined *at the wheels*, the energy at the wheels, E_w, needed to move the vehicle by an amount $x(t)$ is

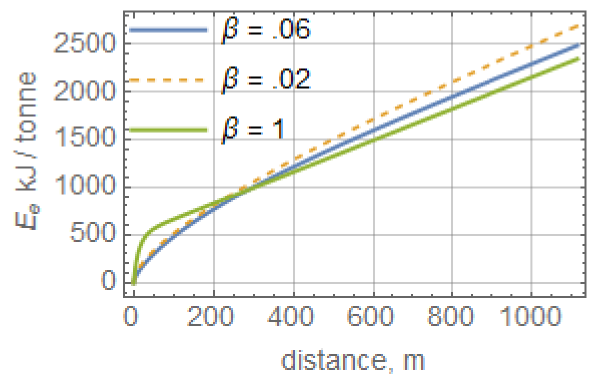
$$E_w(t) = \int_0^t VSP(t)dt \quad \text{or} \quad E_w(x) = \int_0^x VSP(x)/v(x)dx \tag{1.31}$$

in units of kW*sec/tonne= kJ/tonne. The energy *at the engine*, E_e, is greater than E_w because of the energy used during idle periods and the mechanical inefficiencies. For a passenger car we can approximate the VSP at the engine by the linear regression $VSP_e = 3.5235 \times VSP + 14.641$ (which has an $R^2 = 0.98$) and integrate it as in (1.33) to obtain

$$E_e(x) = 3.5235 \times E_w(x) + 14.641 \times t(x)$$

The following figure shows $E_e(x)$ and $E_w(x)$ consumed by our acceleration process for different values of β. It can be seen that the lower the β the higher the energy required at the engine for distances > 300m, and therefore to reduce energy consumption on should accelerate more aggressively.





Emissions

The emissions of a certain pollutant generated by a vehicle trajectory may be expressed in two ways. If the vehicle travels a total distance of X during time T, emissions of pollutant i can be calculated using a **time-defendant description:**

$$\int_0^T f_i(VSP(t))dt \tag{1.32}$$

or a **location-dependent description:**

$$\int_0^X f_i(VSP(x))/v(x)dx \tag{1.33}$$

in units of gr/tonne. The function f_i depends on the pollutant i; as a *crude* approximation, we can use

$$C02: f_1(VSP) = 0.2532 \times VSP + 1.0522 \quad (R^2 = 0.99)$$

$$PM: f_2(VSP) = \exp(0.0425 \times VSP - 5.0915) \quad (R^2 = 0.81)$$

$$NOx: f_3(VSP) = \exp(0.0664 \times VSP - 4.2) \quad (R^2 = 0.96)$$

$$CO: f_4(VSP) = \exp(0.051 \times VSP - 2.664) \quad (R^2 = 0.89)$$

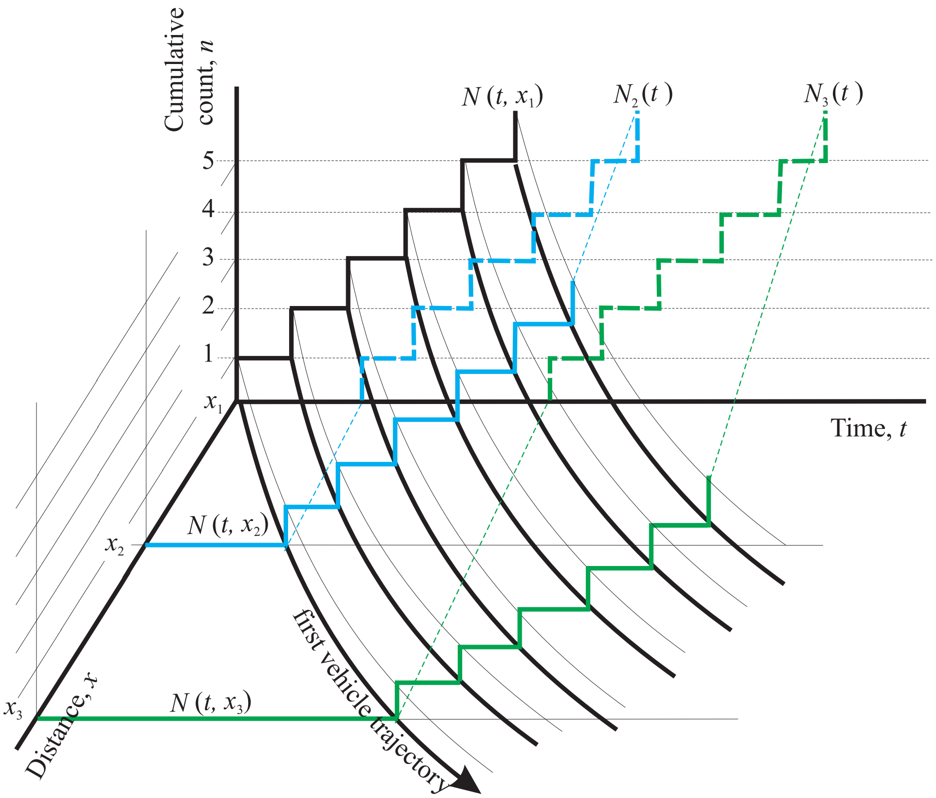
$$THC: f_5(VSP) = \exp(0.0596 \times VSP - 4.8019) \quad (R^2 = 0.9)$$

all in units of gr/tonne.

1.3 Vehicle stream characteristics

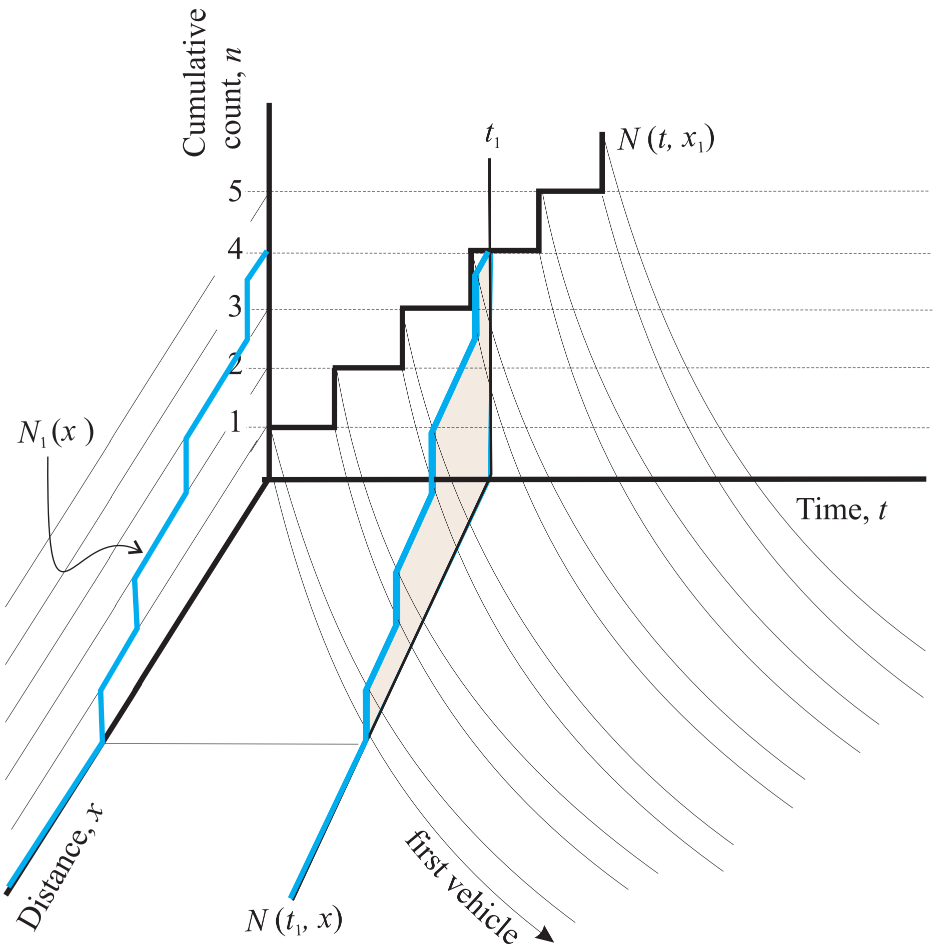
Definition: The traffic flow surface The surface $N(t,x)$, give the cumulative number of vehicles that have crossed location x by time t. It can be measured experimentally by counting vehicles at each location starting with the passage of a reference vehicle. The function $N(t,x)$ is

1. a step function, non-decreasing in time, non-increasing in space,
2. constant along a vehicle trajectory if there is no passing



The curve $N(x, t) = j$ in the (x, t) plane is the contour line of constant cumulative count j. If vehicles do not pass, the j^{th} cumulative arrival at one location is the same vehicle as the j^{th} cumulative arrival at any other location; thus the curve $N(x, t) = j$ is the "trajectory" of the j^{th} car. It describes the time at which the j^{th} car reaches x or equivalently the position $x_j(t)$ of the j^{th} car as a function of time.

Cumulative count curves $N_k(t)$ at a location x_k are the intersection of N and planes $x = x_k$, projected onto the (t, n) space.



The contour lines on the (x, n) plane are not used as commonly in practice as the other two. But if we take an aerial photograph of the highway (fixed time), one would see a succession of vehicles on the x-axis (the highway), and we could draw the cumulative number of vehicles from some arbitrary reference point $x = 0$ to a variable x. The function *N(x, t)* for fixed t is the number of vehicles between the point x and the point further downstream where the reference vehicle labeled 0 is located.

Continuum approximation: **If we smooth the surface** N so as to make it a continuous and differentiable function, then its partial derivatives represent the flow and density at a point :

Definition: Flow and density at a point (t,x)**:**

$$q(t,x) = \partial N(t,x)/\partial t \qquad \text{(flow)} \tag{1.34a}$$

$$k(t,x) = -\partial N(t,x)/\partial x \qquad \text{(density)} \tag{1.34b}$$

The negative sign is because vehicle labels are decreasing with x, but, in absolute value, it is the number of vehicles per unit length of highway.

In the *(x, t)* plane, the contour lines are *trajectories of individual vehicles* and the slope of a contour is the velocity of the vehicles. If one is on a contour line corresponding to $N(t,x) = j$ and one makes as infinitesimal displacement (dx, dt) in the (x, t) plane in such a way as to stay on the contour. Then the (dx, dt) must be such that along a vehicle trajectory: $dN = \frac{\partial N(t,x)}{\partial x}dx + \frac{\partial N(t,x)}{\partial t}dt = 0$, ie:

$$\frac{\partial N(t,x)}{\partial x}\frac{dx}{dt} + \frac{\partial N(t,x)}{\partial t} = 0 \tag{1.35}$$

Since $\mathrm{d}x/\mathrm{d}t = v(x,\ t)$ is the velocity at the point $(x,\ t)$, this can also be written as

Definition: Fundamental traffic flow relationship

$$q = kv \tag{1.36}$$

which is one of the basic identities in continuum traffic flow theory (or fluid mechanics).

1.3.1 Generalized definitions

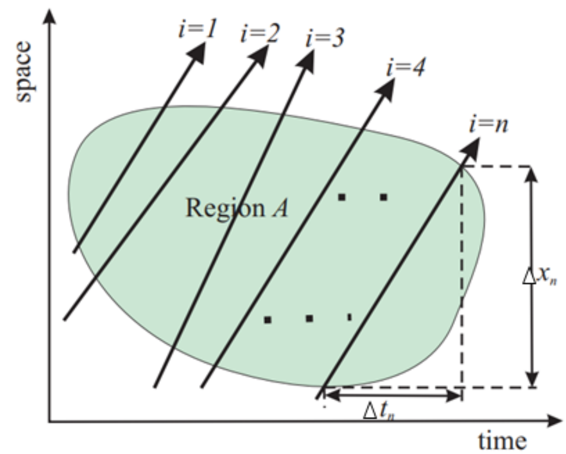
These definitions allow us to compute average traffic variables in any time-space region A using vehicle trajectories $x_i(t)$.

Definition: Edie's generalized definitions The density, flow and speed in A are

$$k(A) = \frac{t(A)}{|A|} = \frac{\sum_{i=1}^{n} \Delta t_i}{|A|} \tag{1.37a}$$

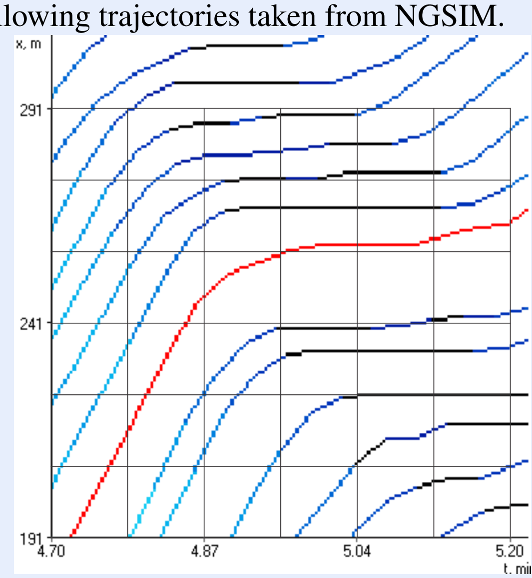
$$q(A) = \frac{d(A)}{|A|} = \frac{\sum_{i=1}^{n} \Delta x_i}{|A|} \tag{1.37b}$$

$$v(A) = \frac{q(A)}{k(A)} = \frac{d(A)}{t(A)}, \tag{1.37c}$$



A is any time-space region, $|A|$ = 'area' of A (mile-hrs)
$t(A)$ = total travel time in A (veh-hrs) $d(A)$ = total travel distance in region A (mile-veh)

Example 1.10. Use Edie's generalized definitions to compute the average traffic flow variables of the following trajectories taken from NGSIM.



Ans: v = 8.1 km/hr, k = 76.6 veh/km, and q = 616.9 veh/hr
(with the Trajectory Explorer application) ■

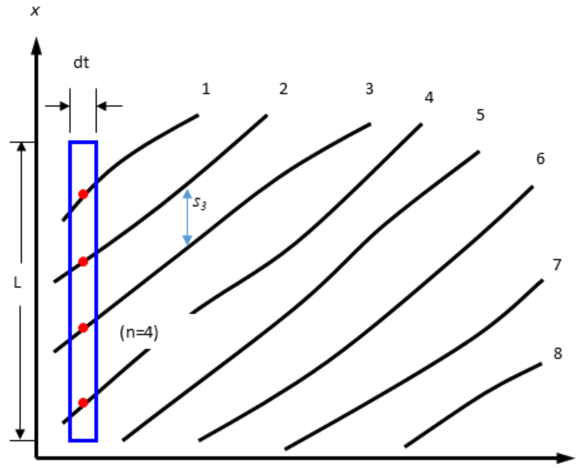
Instantaneous measurements

From an instantaneous photo at a given time t_0 we have:

$$|A| = Ldt, \quad t(A) = n \cdot dt \quad \text{and} \quad d(A) = \sum_{i=1}^{n} v_i \, dt$$

and therefore

$$k = n/L, \quad q = \frac{1}{L}\sum_{i=1}^{n} v_i \quad \text{and} \quad v = \frac{1}{n}\sum_{i=1}^{n} v_i \tag{1.38}$$



Definition: Spacing: $s_i(t)$ is the distance between vehicle i and its leader $i-1$ at time t: $s_i(t) = x_{i-1}(t) - x_i(t)$. The density is the inverse of the average spacing. Since $L \approx \sum_{i=1}^{n} s_i$:

$$k = \frac{n}{\sum_{i=1}^{n} s_i} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} s_i} = 1/\bar{s}$$

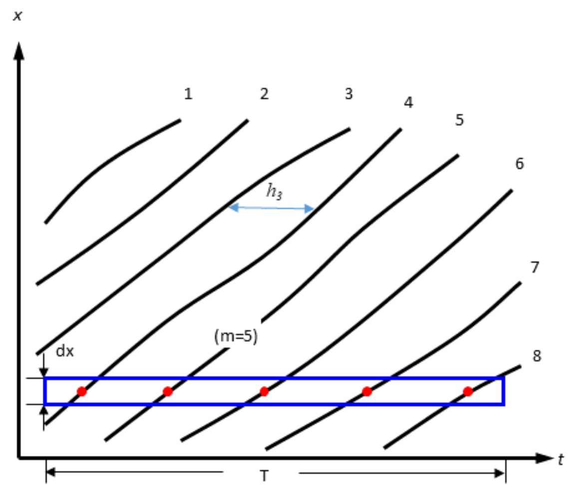
Stationary measurements

At a fixed location x_0 we have:

$$|A| = Tdx, \quad t(A) = \sum_{i=1}^{m} dx/v_i \quad \text{and} \quad d(A) = m \cdot dx$$

and therefore

$$q = m/T, \quad k = \frac{1}{T}\sum_{i=1}^{m} \frac{1}{v_i} \quad \text{and} \quad v = \frac{1}{\frac{1}{m}\sum_{i=1}^{m} \frac{1}{v_i}} \tag{1.39}$$



Definition: Headway: $h_i(x)$ is the time between the passage of vehicle i and $i+1$ at x. The flow is the inverse of the average headway. Since $T \approx \sum_{i=1}^{m} h_i$:

$$q = \frac{m}{\sum_{i=1}^{m} h_i} = \frac{1}{\frac{1}{m}\sum_{i=1}^{m} h_i} = 1/\bar{h}$$

Example 1.11. — Moving observer Show that if an observer (eg, a drone) is moving at a constant speed v_0 for a duration T recording the speed v_i of the m of vehicles encountered, then

$$q = \frac{1}{T}\sum_{i=1}^{m} \frac{v_i}{|v_i - v_0|}, \quad k = \frac{1}{T}\sum_{i=1}^{m} \frac{1}{|v_i - v_0|} \quad \text{and} \quad v = \frac{\sum_{i=1}^{m} \frac{v_i}{|v_i - v_0|}}{\sum_{i=1}^{m} \frac{1}{|v_i - v_0|}} \tag{1.40}$$

→ Solution ■

Definition: Stationary conditions arise whenever traffic variables are independent of time and space. Therefore, vehicle trajectories look like evenly spaced and parallel straight lines, ie:
$v_1 = v_2 = \cdots = v$
$s_1 = s_2 = \cdots = s$
$h_1 = h_2 = \cdots = h$
Note: $v = s/h = q/k$.

Definition: The passing rate r_0 of a moving observer of speed v_0 is the number of vehicles in the traffic stream that *pass the observer*. If conditions are stationary with constant q, k, v then

$$\boxed{r_0 = k(v - v_0) = q - kv_0} \qquad \text{(Moving observer formula)} \tag{1.41}$$

Notice that when $v_0 > v$ then the passing rate is negative, which means that the observer is the one passing other vehicles.

Example 1.12. In the example 1.11 show that if conditions are stationary with traffic variables q, k, v then the moving observer formula (1.41) follows. ■

1.3.2 Families of vehicles

Imagine we breakdown the overall traffic flow into different groups of vehicles, or families, that appear to be stationary. For example, vehicle type, vehicles on different lanes, different roads, different origin or destination, etc.

Since we assume that each family $i = 1, 2\ldots$ is in stationary conditions traveling at a constant speed v_i, we have:

$$q_i = k_i v_i \tag{1.42}$$

and the total flow and density are:

$$q = \sum_i q_i \quad \text{and} \quad k = \sum_i k_i \tag{1.43}$$

The question: what is the average speed v that is consistent with the fundamental traffic flow relationship ?

$$q = kv$$

Definition: The time-mean and space-mean proportions are:

$$p_i^t = \frac{q_i}{q} \quad \text{and} \quad p_i^s = \frac{k_i}{k} \tag{1.44}$$

and can be interpreted as probability distributions.

Using (1.42) and (1.43) we obtain

$$q = k \sum_i p_i^s v_i \quad \text{and} \quad k = q \sum_i p_i^t \frac{1}{v_i} \tag{1.45}$$

The average speed that is consistent with $q = kv$ is $\sum_i p_i^s v_i$, or $\left(\sum_i p_i^t \frac{1}{v_i}\right)^{-1}$ which is called the space-mean speed $\overline{v}_s$:

$$\boxed{\overline{v}_s = \sum_i p_i^s v_i = \left(\sum_i p_i^t \frac{1}{v_i}\right)^{-1}} \qquad \text{(space-mean speed)} \tag{1.46}$$

The time mean speed

$$\overline{v}_t = \sum_i p_i^t v_i \tag{1.47}$$

should never be used because it is not consistent with the fundamental traffic flow relationship. In fact:

$$\overline{v}_t \geq \overline{v}_s \tag{1.48}$$

which will be proven in the next section.

R For a measurement of duration T at a fixed location

$$p_i^t = \frac{q_i}{q} = \frac{m_i/T}{m/T} = \frac{m_i}{m} \tag{1.49}$$

where m_i is the count for family i during T, and $m = \sum m_i$. In addition, if each vehicle is its own family, $m_i = 1$ and

$$p_i^t = \frac{1}{m} \tag{1.50}$$

then (1.46) becomes

$$\overline{v}_s = \left(\frac{1}{m}\sum_i \frac{1}{v_i}\right)^{-1} \tag{1.51}$$

which coincides with (1.39), as expected.

Example 1.13. — 4 cars on a circular 1 km road travel at constant speeds of 20, 40, 60 and 80 km/hr, respectively. Calculate the space- and time-mean speeds. ■

Example 1.14. — Cars and trucks. The flows of cars and trucks on a very long upgrade are 10 and 2 vehicles per minute respectively. Trucks travel at 0.8 mile/min and cars at 1.0 mile/min when not trapped behind a truck (otherwise they travel at 0.8 mile/min). If the time used to climb a 1 mile section of the grade averaged across all the vehicles observed over a long time is 1.1 min. Determine:

1. The density of vehicles on the upgrade.
2. The average speed on the grade, taken across cars.
3. The proportion of time that each car spends behind trucks.

■

i	description	v_i	q_i	k_i
1	Fast cars	1	$(1-\alpha)k_C\, v_1$	$(1-\alpha)k_C$
2	Cars behind trucks	0.8	$\alpha k_C\, v_2$	αk_C
3	Trucks	0.8	2	2.5
TOT		q		k

We have $q_C = 10$ and the space-mean speed is $\bar{v}_s = 1/1.1$. Let:

$$k_C = k_1 + k_2, \qquad q_C = q_1 + q_2, \qquad \alpha = k_2/k_C$$

Solving the following system of equations gives $\alpha = 0.327, k_C = 10.7$:

$$\begin{cases} k\bar{v}_s = q_C + q_3 & (1.52a) \\ q_C = (1-\alpha)k_C\, v_1 + \alpha k_C\, v_2 & (1.52b) \end{cases}$$

1. The density of vehicles on the upgrade: $k = 13.2$
2. The average speed on the grade, taken across cars: $v_C = q_C/k_C = (1-\alpha)v_1 + \alpha\, v_2 = 0.9345$
3. The proportion of time that each car spends behind trucks: $\alpha = 0.327$ (because of previous answer)

Average of other variables

Let r be an arbitrary quantity of interest, such as vehicle weight, vehicle occupancy (persons per vehicle), emissions, age of the vehicle or any other *random variable* of interest. We let r_i be the value of this quantity for family i, which implies that the probability distribution of r_i is given by the proportions p_i^s. The averaging of this characteristic across families, $\mathrm{E}[r]$, should be done spatially

$$\boxed{\mathrm{E}_s[r] = \sum r_i\, p_i^s} \qquad (1.53)$$

rather than temporally

$$\mathrm{E}_t[r] = \sum r_i\, p_i^t \qquad (1.54)$$

as seen on the previous section. [1] If all we have are the time-average proportions we can still compute the correct averages. Using the traffic flow relationship in (1.43) we get

$$p_i^t = p_i^s \frac{v_i}{\bar{v}_s} \qquad (1.55)$$

and therefore

$$\mathrm{E}_s[r] = \sum r_i\, p_i^t \frac{\bar{v}_s}{v_i}$$

To see the bias when using time-averages instead of space-averages, we rewrite (1.54) using (1.55) to obtain $\mathrm{E}_t[r] = \frac{1}{\bar{v}_s}\sum r_i\, v_i\, p_i^s = \frac{1}{\bar{v}_s}\mathrm{E}_s[r\cdot v]$. Since $\mathrm{E}_s[r\cdot v] = \mathrm{E}_s[r]\,\bar{v}_s + \mathrm{Cov}_s[r,v]$, the bias is [2]

$$\boxed{\mathrm{E}_t[r] - \mathrm{E}_s[r] = \frac{\mathrm{Cov}_s[r,v]}{\bar{v}_s}} \qquad (1.56)$$

[1] Using this notation, notice that:

$$\mathrm{E}_s[v] = \bar{v}_s$$

[2] Recall that for a random variables X, Y we have that $\mathrm{Cov}_s[X,Y] = E[XY] - E[X]E[Y]$.

Therefore, if r and v are:

- positively correlated, then $\mathrm{E}_t[r] > \mathrm{E}_s[r]$
- negatively correlated, then $\mathrm{E}_t[r] < \mathrm{E}_s[r]$
- uncorrelated, then $\mathrm{E}_t[r] = \mathrm{E}_s[r]$

Example 1.15. For the following quantities indicate the sign and magnitude of the bias $\mathrm{E}_t[r] - \mathrm{E}_s[r]$:

a) vehicle weight
b) vehicle length
c) vehicle emissions
d) age vehicles
e) speed
f) trip length
g) vehicle occupancy across HOV and general purpose lanes

If the characteristic of interests happens to be the speed, i.e. $r = v$, then $\mathrm{E}_t[v] = \mathrm{E}_s[v] + \frac{\mathrm{Var}_s[v]}{\bar{v}_s}$, or:

$$\bar{v}_t = \bar{v}_s + \frac{\mathrm{Var}_s[v]}{\bar{v}_s} \tag{1.57}$$

which means that $\bar{v}_t \geq \bar{v}_s$ because the variance is always nonnegative.

What about the variance of r? (optional)

By definition:

$$\mathrm{Var}_s[r] = \mathrm{E}_s\left[r^2\right] - \mathrm{E}_s[r]^2, \tag{1.58a}$$

$$\mathrm{Var}_t[r] = \mathrm{E}_t\left[r^2\right] - \mathrm{E}_t[r]^2, \tag{1.58b}$$

Combining this with (1.56) it can be shown that:

$$\mathrm{Var}_t[r] - \mathrm{Var}_s[r] = \frac{\mathrm{Cov}_s\left[r^2, v\right]}{\bar{v}_s} - \left(\frac{\mathrm{Cov}_s[r, v]}{\bar{v}_s}\right)^2 - \frac{2\mathrm{E}_s[r]\,\mathrm{Cov}_s[r, v]}{\bar{v}_s} \tag{1.59}$$

which may be positive or negative.

For linear functions $r(v) = a + bv$ one can show that

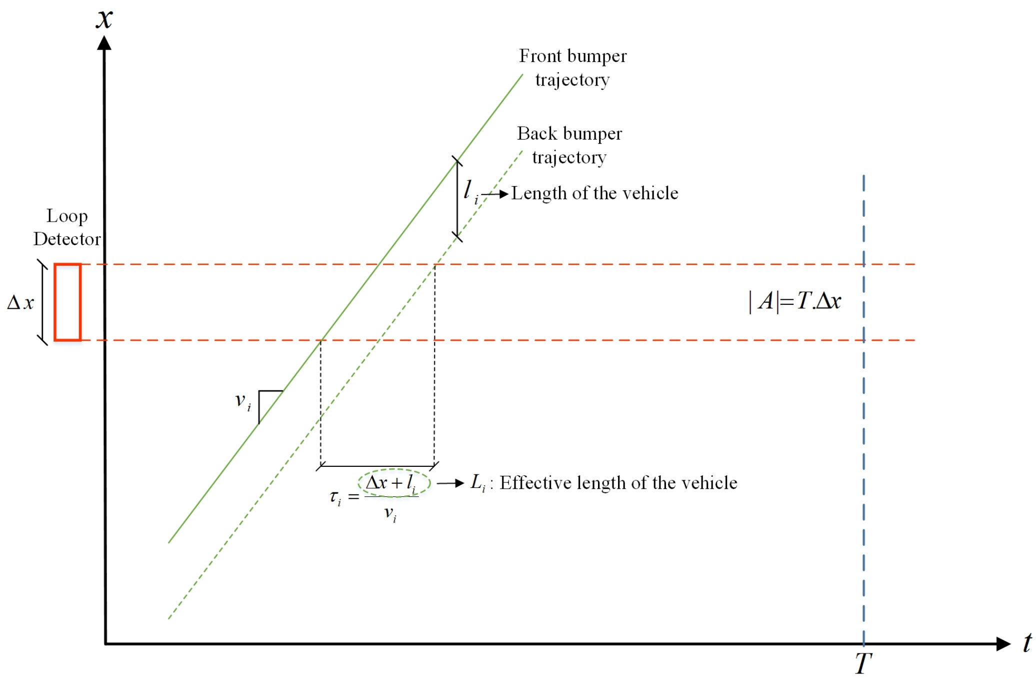
$$\mathrm{Var}_t[r] - \mathrm{Var}_s[r] = b^2\left(\frac{\mathrm{E}_s\left[v^3\right]}{\bar{v}_s} - \mathrm{Var}_s[r]\,(3 + \delta_v^2) - \bar{v}_s^2\right) \tag{1.60}$$

where $\delta_v^2 = \mathrm{Var}_s[v]/\mathrm{E}_s[v]^2$ is the squared coefficient of variation of v, typically $\ll 1$. The sign of the bias depends on the (space-mean) distribution of the speed:

PDF of v	Bias, eqn. (1.60)
Exponential $[1/v_s]$	$b^2 v_s^2$
Poisson $[v_s]$	0
Uniform $[\{v_s-\sigma, v_s+\sigma\}]$	$-b^2\sigma^4/(9v_s^2)$
Normal $[v_s, \sigma]$	$-b^2\sigma^4/v_s^2$
LogNormal $\left[\log(v_s)-\frac{\sigma^2}{2}, \sigma\right]$	$b^2\left(e^{\sigma^2}-1\right)^2\left(e^{\sigma^2}+1\right)v_s^2$
Binomial $[v_s/p, p]$	$b^2(p-1)p$

As can be seen, the sign of the bias is positive except for the normal and uniform distributions.

Application: occupancy from loop detectors.



Loop detectors can easily measure flow and speed, but cannot measure the density directly. Instead, they can measure a proxy for the density. The occupancy, o, is the proportion of time a vehicle is on top of the loop detector:

$$o = \frac{\sum_i^m \tau_i}{T} = \frac{1}{T}\sum_i L_i/v_i \tag{1.61}$$

To see the relationship with the density, recall eqn. (1.39) that at a fixed location the generalized definition of density gives

$$k = \frac{1}{T}\sum_{i=1} \frac{1}{v_i}$$

If all vehicle lengths are identical and equal to L, combining these two equations gives

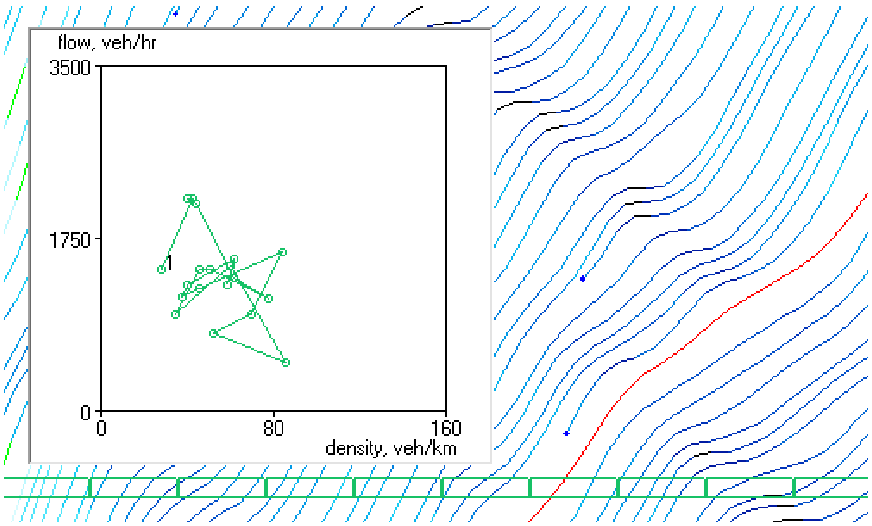
$$o = Lk$$

but in reality this assumption is not accurate, and one should replace L by its expected value to obtain:

$$\boxed{o = \mathrm{E}_s[L]\,k} \tag{1.62}$$

and using eqns. (1.50), (1.55), and (1.51):

$$\mathrm{E}_s[L] = \bar{v}_s \frac{1}{m}\sum_{i=1}^{m}\frac{L_i}{v_i} = \frac{\frac{1}{m}\sum_i \frac{L_i}{v_i}}{\frac{1}{m}\sum_i \frac{1}{v_i}} = \frac{\sum_i \frac{L_i}{v_i}}{\sum_i \frac{1}{v_i}}$$



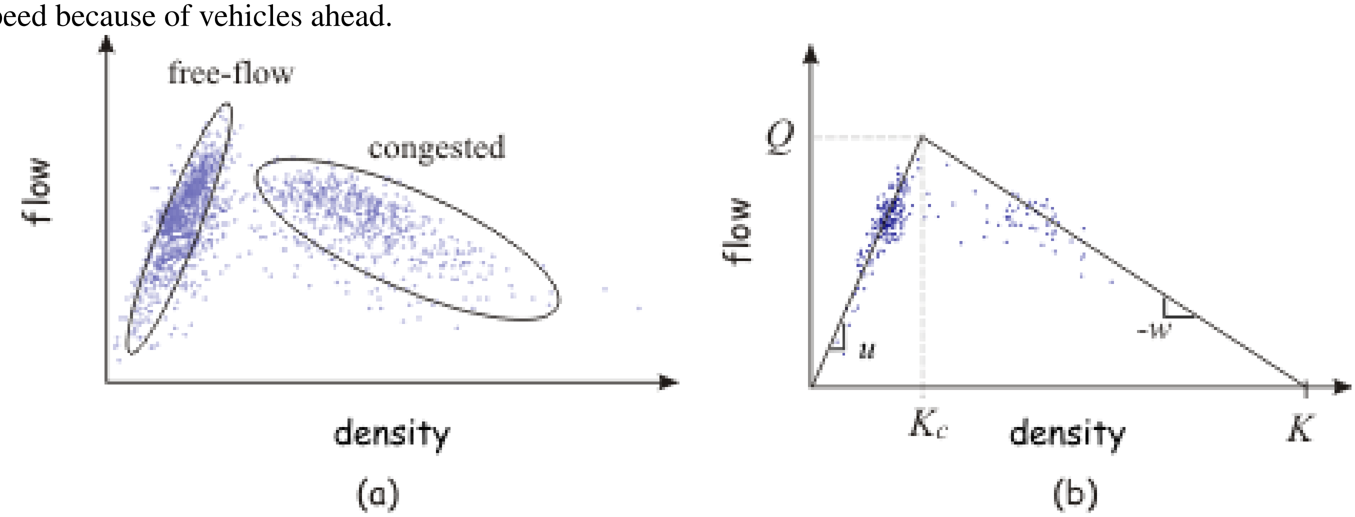
The above figure shows a typical flow-density scatterplot under congested traffic. They typically exhibit a large scatter because vehicles inside each region of measurements may not be in stationary conditions.

1.3.3 Fundamental Diagrams

A fundamental diagram (FD) is a function that relates any 2 traffic flow variables (such as q, k, v, h, s). One of the most useful is the flow-density FD:

$$q = F(k) \tag{1.63}$$

Traffic states can be grouped in two *phases* or *regimes*: (i) free-flow regime, given by the increasing portion of the flow-density FD, where vehicles do not interact very much; (ii) congested or queued regime, given by the decreasing portion of the flow-density FD, where drivers are not able to travel at their desired speed because of vehicles ahead.



R The kinematic wave model, the main subject of this textbook, is (1.63) combined with the definitions of flow and density as a function of cumulative counts $N(t,x)$:

$$\frac{\partial N}{\partial t} = F(-\frac{\partial N}{\partial x})$$

This is a Hamilton-Jacobi PDE and will be analyzed in detail in chapter 4.

Definition: Triangular flow-density fundamental diagram A triangular flow-density fundamental diagram may be defined by

1. free-flow speed, u
2. wave speed, $-w$,
3. jam density K.

Other useful resulting parameters that will be used are:

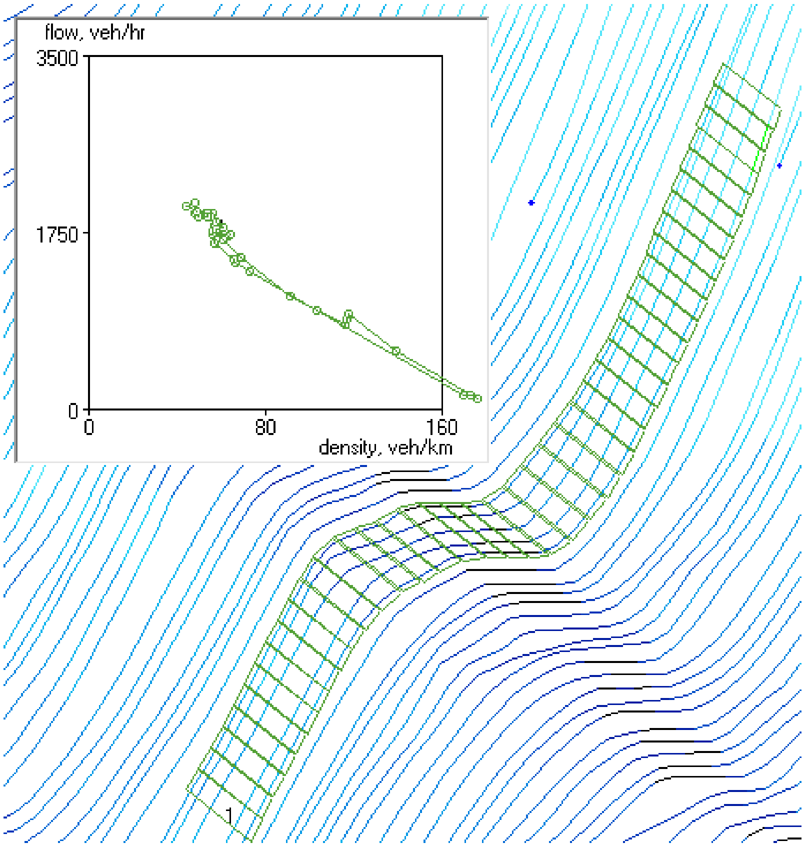
- capacity, $Q = Kwu/(w+u)$,
- critical density $K_c = Q/u$,
- critical spacing $s^* = 1/K_c$,
- jam spacing $\delta = 1/K$, and
- wave trip time between two consecutive vehicles, $\tau = 1/(wK)$.

Triangular FD's are the simplest ones consistent with empirical data.

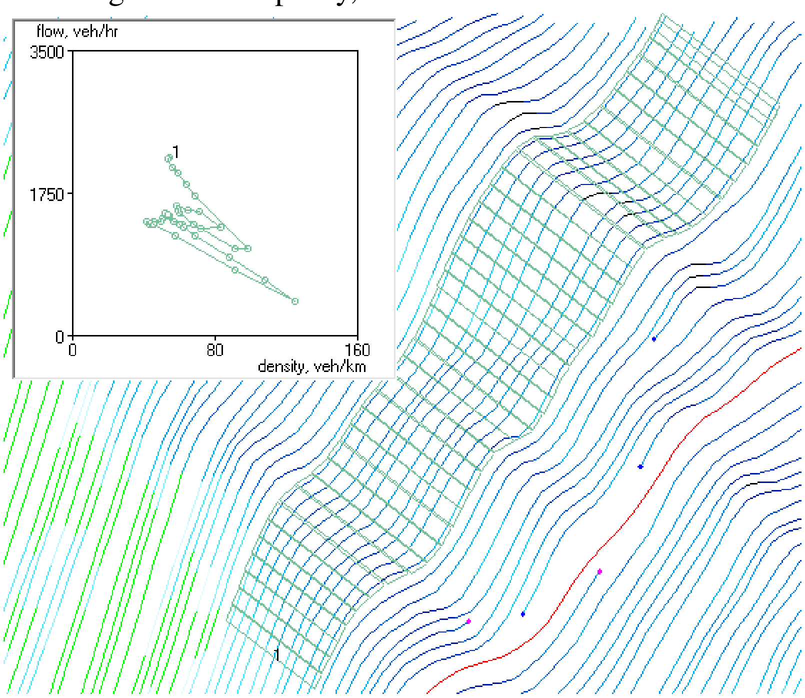
The scatter and appearance of FD's obtained in the field depends on the following factors

a) the aggregation interval and on the occurrence of regime changes.The dots in the figure abovedepict a typical scatter plot of loop-detector flow-occupancy data for all lanes on consecutive 30-sec intervals. If 3-min intervals are used, the plot becomes less scattered as shown in part (b) of the figure. Note how well a triangular FD fits the data.
b) If traffic is not stationary, measurements will tend to fall inside the FD. This problem is exacerbated when the measurement aggregates all lanes, since different lanes might be in different regimes.
c) as illustrated in the figure below, the appearance of the FD depends on the location relative to the prevailing bottleneck.

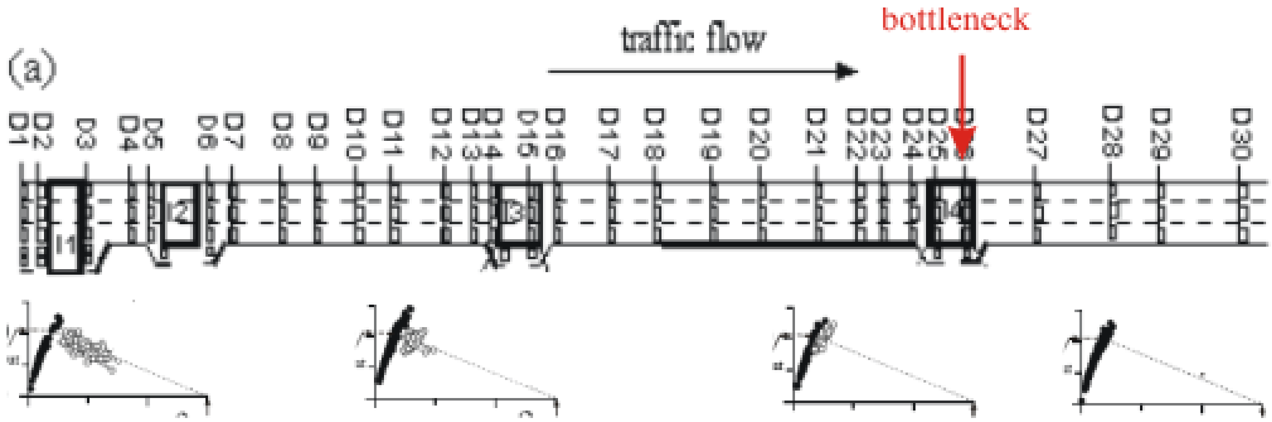
The figure below shows that scatter can be minimized by choosing regions where vehicles are in stationary conditions.



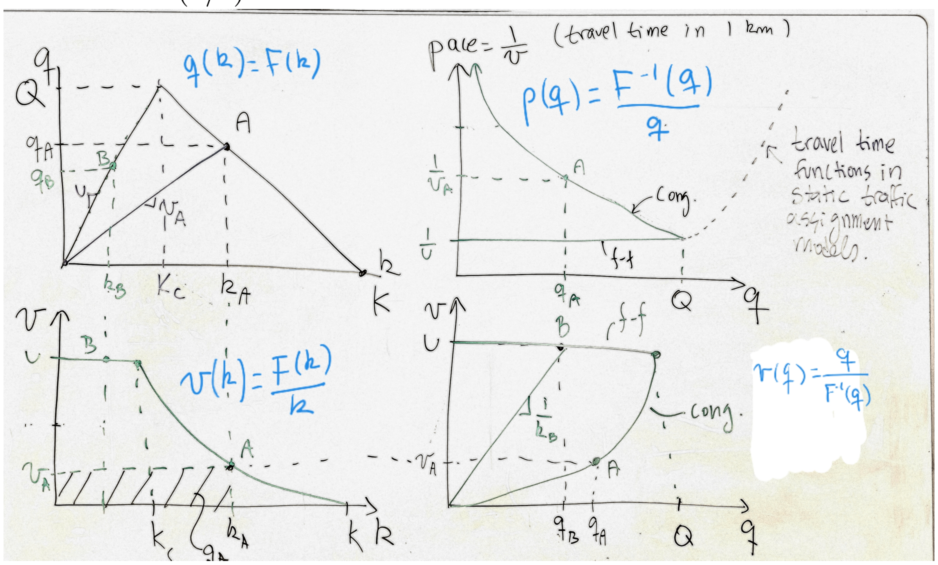
But for some group of drivers scatter cannot be eliminated because some of them change the way they drive coming out of a stop way, as illustrated below.



Finally, the appearance of empirical FDs depends on *where* we measure:



Other fundamental diagrams can be derived combining (1.63) with the fundamental traffic flow relationship. For example, the speed-density FD is given by $v = q/k = F(k)/k$ and the speed-spacing FD would be $v = s\,F(1/s)$.



The pace-flow FD and static traffic assignment

The pace, $p = 1/v$ is the inverse of speed, and can be interpreted as the travel time over one distance unit. As shown in the above figure, the pace-flow relationship consistent with traffic flow is very different from the travel time-flow functions used in static traffic assignment models. This means that static assignment models, which are at the core of the wildly used four-step travel demand models, are unable to capture congestion properly: they allow flows larger than capacity on the links, which is physically impossible, and in which case they predict travel times that are arbitrary.

1.3.4 The conservation equation

Definition: For a road without entrances or exists the conservation law is:

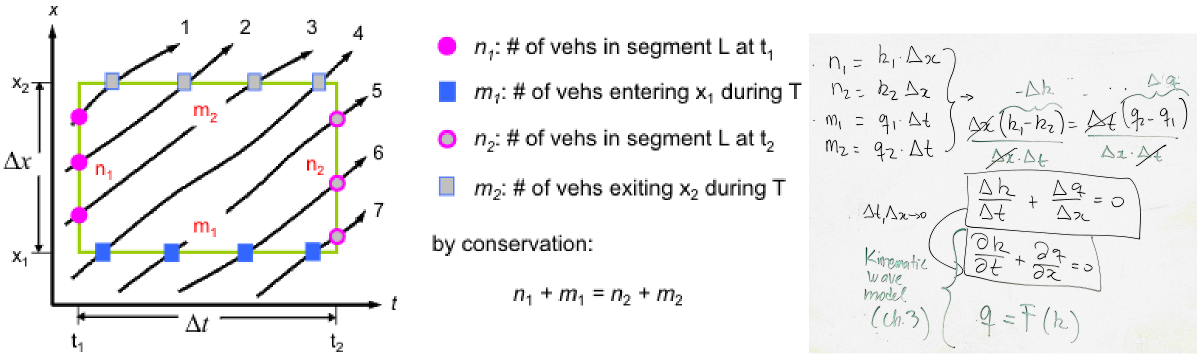
$$\frac{\partial k(t,x)}{\partial t} + \frac{\partial q(t,x)}{\partial x} = 0. \tag{1.64}$$

PDE (1.64) is first-order, nonlinear and hyperbolic. It can also be written as

$$\frac{\partial k}{\partial t} + \frac{\partial (v \cdot k)}{\partial x} = 0$$

Kinematic wave theory is (also) the combination of the conservation law and the fundamental diagram.

First derivation



Second derivation: symmetry of second derivatives

Any function having differentiable partial derivatives, such as *N(x, t)*, satisfies

$$\frac{\partial^2 N(t,x)}{\partial x \partial t} = \frac{\partial^2 N(t,x)}{\partial t \partial x}$$

This gives the conservation law (1.64) recalling the definitions of flow and density at a point, eqn (1.34).

Third derivation

Greens' theorem If L and M are functions of (t,x) defined on an open region A bounded by C and have continuous partial derivatives there, then

$$\oint_C (L dt + M dx) = \iint_A \left(\frac{\partial M}{\partial t} - \frac{\partial L}{\partial x} \right) dt\, dx$$

where the path of integration along C is *anticlockwise*. → More info...

Letting $L = q$ and $M = -k$ in Greens' theorem we obtain

$$\oint_C (q dt - k dx) = -\iint_A \left(\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} \right) dt\, dx \tag{1.65}$$

Given definition (1.34), the vector $(q, -k)$ is the gradient of $N(t,x)$, and therefore

$$\oint_C (qdt - kdx) = N(t_2, x_2) - N(t_1, x_1)$$

is the difference in vehicle number between two points, 1 and 2, and is independent of the path C between the two points. When this path C is closed, such as the perimeter of traffic region A,

$$\oint_C (qdt - kdx) = 0 \qquad \text{(when } C \text{ is closed)}$$

and the conservation law (1.64) follows because (1.65) is valid for all regions A. In the figure above, $\oint_C q dt = m_1 - m_2$ and $\oint_C k dx = n_2 - n_1$ which gives $n_1 + m_1 = n_2 + m_2$, as expected.

Example 1.16. — Moving Observer Find the passing rate r_0 measured by moving observer traveling at constant speed v_0 on a road in stationary traffic state (q,k).

Here:

$$N(t_2,x_2) - N(t_1,x_1) = \oint_C q dt - k dx = qT - kL$$

dividing by T gives

$$r_0 = q - kv_0 \tag{1.66}$$

■

Example 1.17. From two consecutive film frames of stationary traffic along some road, one observes that there are 20 cars/km traveling at 100 km/hr and 30 cars/km traveling at 120 km/hr.

a) If individual cars can maintain these speeds even while passing other cars, how many cars will one of the drivers traveling at 120 km/hr pass while traveling one kilometer? (ans. 3.33 cars)

b) How many passing maneuvers are executed by all drivers in 1 km of road during one hour? (ans. $12,000$ maneuvers)

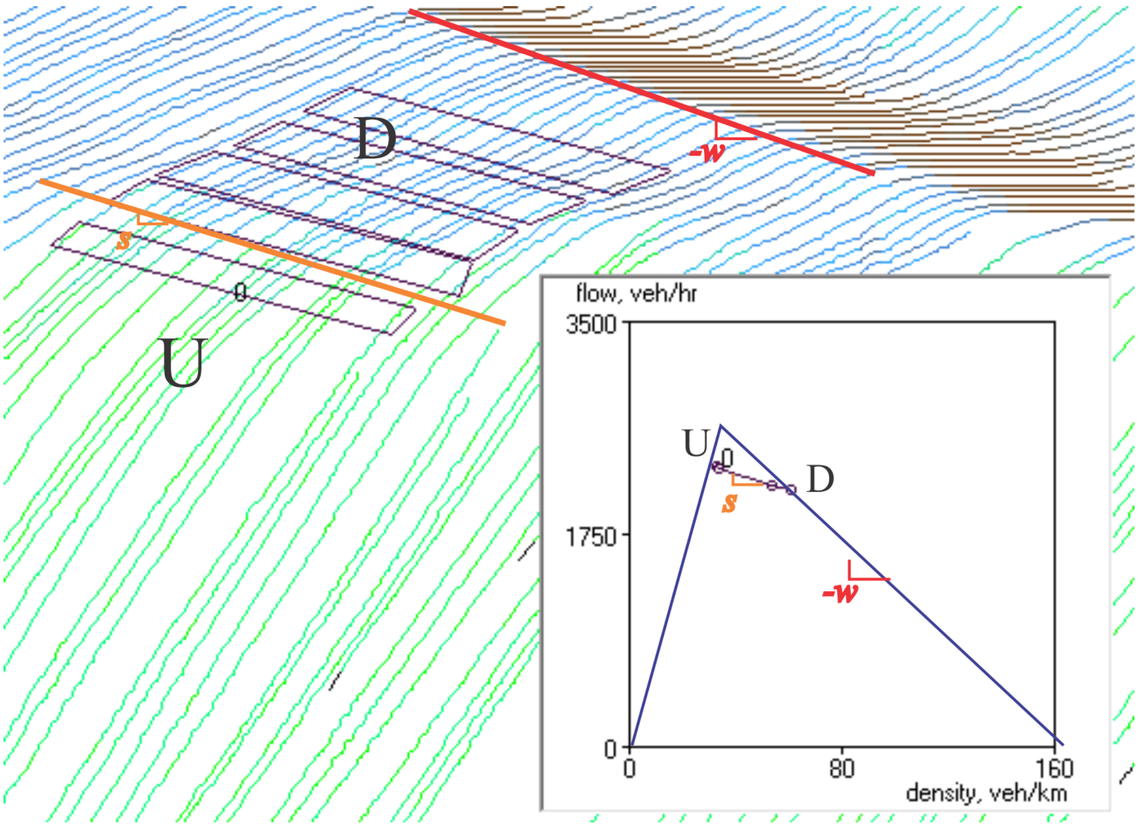
■

Example 1.18. — Speed of interfaces Two neighboring time-space regions are in stationary states $U = (q_U, k_U)$ and $D = (q_D, k_D)$, respectively. Show that the interface separating these traffic states has a slope of

$$s = \frac{q_U - q_D}{k_U - k_D}$$

R In the real world this can be observed whenever $v_U > v_D$, i.e. when decelerating to join the back of the queue (BOQ). When $v_U < v_D$ we drive in such a way that $s = w$.

■



1.4 Problems

Problem 1.1 — Three friends and one bicycle* Three friends, A, B and C, are traveling a very long distance on a bike that only seats two. The strategy they use is as follows. Friend C starts walking while A and B ride the bike; then, after some time, say T, friend B gets off the bike and continues walking while A goes back to pick up C and then both ride back to meet with B. At this point, the cycle starts over. The walking speed is V_0, the bicycle speed with two people is V_2 and with one person is V_1.

1. Neglecting the u-turn time (ie, the time it takes to turn the bike around plus the time the 2^{nd} bike passenger takes to board/alight the bike), determine whether or not the average speed of the three friends is independent of T.
2. Determine the average speed if V_0 = 3 mph, V_1= 9 mph and V_2 = 6 mph.
3. What is the optimal strategy if one cannot neglect the u-turn time? explain.

Problem 1.2 — Tram and car crash? A car and a tram traveling perpendicularly to each other break simultaneously in order to avoid collision. Both drivers have reaction times of one sec. The distance of the tram to the potential collision point is 50 m, while that of the car is 30 m. The speed of the tram is 40 km/hr, and the maximum possible deceleration is -2.1 m/s^2. The corresponding values for the car are 60 km/hr and -8 m/s^2. Will the two collide?

Problem 1.3 — Distance-dependent motion Consider a vehicle moving with a distance-dependent acceleration given by:

$$a(x) = e^{-x}$$

Show that if $v_0 = 0$:

$$\begin{aligned} v(x) &= \sqrt{2(1-e^{-x})} \\ t(x) &= \sqrt{2}\log\left(\sqrt{2(e^x-1)}+\sqrt{2}e^{x/2}\right), \end{aligned}$$

and derive the time-dependent trajectory and speed.

Problem 1.4 — Distance-dependent vehicle kinematics* Consider a vehicle moving with an acceleration given by:

$$a(v,x) = 1 - v - x/4$$

Show that if $t_0 = x_0 = v_0 = 0$ then:

$$\begin{aligned} v(t) &= te^{-t/2} \\ x(t) &= 4 - 2e^{-t/2}(t+2) \\ a(t) &= \frac{1}{2}e^{-t/2}(2-t) \end{aligned}$$

Problem 1.5 — Effects of curvature on vehicle kinematics Investigate how to include the curvature of the roadway on a vehicle kinematics model and exemplify with an Euler-type simulation of a vehicle taking a "cloverleaf" off-ramp.

Problem 1.6 — Car accelerates upstream of platoon A car is stopped at an entrance ramp to a freeway; its driver is preparing to merge. At a certain moment while stopped, this driver observes a platoon of vehicles a distance X_o upstream. The platoon approaches the entrance ramp at a constant speed V. The stopped car can accelerate from speed 0 to V according to the vehicle kinematics model a(v) = p/v, where v is the speed of the car and p the power of the engine.

a) In the t-x diagram, illustrate the "latest safe start time", T, defined as the latest time at which the stopped car can safely start the merge maneuver. Assume this latest time enables the platoon to maintain its constant speed and "just touch" the merging car's trajectory. Ignore the physical dimensions of the car.
b) Derive an expression for T in terms of the variables given and evaluate for X_o, = 500 m, V = 120 km/hr and p= 20 m^2/s^3.
c) How does b) change if there is a 4% upgrade? You may need a numerical solution method here.
d) BONUS: derive a parameter-free version of this vehicle kinematics model for road segment with 100G percent grade, and give expressions for the location, speed, acceleration and jerk as a function of time.

Problem 1.7 — Eco-driving* Using example (1.2.6) as a starting point, derive the eco-driving strategy that minimizes the emissions of one of the following pollutants: particular matter (PM), NOx, CO and hydrocarbons (THC), for a single acceleration process of a typical passenger car.

Problem 1.8 — Train line capacity Consider commuter trains serving contiguous stations. At each station, trains stop to load and unload passengers and this requires a fixed "dwell" time, d. Between stations, trains travel at cruise speed, v. We ignore the effects of acceleration and deceleration. For safety reasons, consecutive trains must always be physically separated by a minimum block distance, b, where block distance is measured from the rear end of a leading train to the front end of the following train. Trains have physical length L.

DETERMINE the maximum flow of trains possible given the above operating conditions. Express this maximum flow in terms of the variables given.

Problem 1.9 — Vehicle occupancy averages* Consider the following loop-detector data below for ten 15-min time intervals (Spreadsheet here). Lane 1 is an HOV lane where it has been observed that the average vehicle occupancy during a 15-min time interval follows a normal distribution with mean 3 passengers per vehicle and a coefficient of variation of 20%. For the general-purpose lanes you can assume a (deterministic) vehicle occupancy of one passenger per vehicle.

a) Compute the time-mean speed and the space-mean speed for all lanes combined
b) For each time interval, generate a vehicle occupancy for the HOV lane, and compute both the time-mean and space-mean vehicle occupancy across lanes.

c) Explain the discrepancies of both aggregation methods (time-mean and space-mean).

SPEED - mi\hr				
time Inte	lane 1	lane 2	lane 3	lane 4
1	47	26	32	38
2	67	27	29	26
3	70	27	27	22
4	72	42	36	24
5	80	35	34	33
6	69	37	31	26
7	46	28	26	28
8	41	40	25	27
9	43	29	40	29
10	44	24	24	31

FLOW - veh in 15 min					
time In	lane 1	lane 2	lane 3	lane 4	q
1	500	396	444	476	1816
2	418	402	425	394	1639
3	397	405	408	358	1570
4	378	493	468	379	1718
5	278	464	458	453	1654
6	402	472	437	393	1704
7	500	416	390	409	1715
8	489	485	388	407	1770
9	494	424	486	419	1823
10	496	376	375	434	1681

Problem 1.10 From consecutive aerial photos of traffic along a highway, one observes that there are ten cars per kilometer with 0 velocity (they are parked), forty cars per kilometer traveling at 15 km/hr and twenty cars per kilometer traveling at 30 km/hr.
a. Determine the space mean speed of traffic and the time mean speed that would be measured at a point on the highway.
b. Under what condition(s) are measured values of time mean speed and space mean speed equal?
c. If individual cars maintain their speeds even while over-taking other cars, how many cars (including parked cars) will a driver traveling at 30 km/hr over-take while traveling one kilometer?

Problem 1.11 From an aerial photograph, one observes that on a level section of a (multilane) highway, 25% of the vehicles are trucks, 75% are cars, and there are 50 vehicles per mile of highway. The trucks travel at 40 mph and the cars at 60 mph. This highway also has a section with a steep grade on which the speed of the trucks drops to 25 mph and the speed of the cars to 55 mph. No vehicles enter or exit the subject highway sections (except at the ends). The flows on these highway sections are stationary, which means that the flows on the level section are equal to those on the grade. Determine

1. the flow of vehicles on the level section;
2. the density of vehicles on the grade;
3. the percent of trucks on the grade as seen from an aerial photo;
4. the percent of trucks as seen by a stationary observer on the grade.

Problem 1.12 A one way road is shared by buses and cars. Buses travel at speed v' and carry n' people. Cars travel at speed v and carry n people. The fraction of vehicles that are buses passing a stationary observer is p. Answer the following questions:
(a) What fraction of the vehicles on a photograph would be buses?
(b) What is the average vehicle occupancy seen by a stationary observer?
(c) What is the average vehicle occupancy seen on a photograph? Evaluate both occupancy averages for v' = 50 km/hr, v = 80 km/hr, n = 1, n' =20 and p= 0.2.
(d) Consider the numerical data of part (c). If a bus emits 2.5 times the pollutants emitted by a car per unit time, what fraction of the total pollution is generated by the cars along one kilometer of road in one hour?

Problem 1.13 — Fundamental diagrams* Suppose that on a freeway the flow-density curve is triangular in shape with parameters: free-flow speed u = 120 km/hr, jam density K = 490 veh/km, and critical density kc = K/7 veh/km.

a) show the equations and sketch the relations between: k and v, k and q, v and q, h and p (pace), and v and s.

b) Assume that a speed limit of $\alpha \cdot u$ is imposed. Using your sketches from part **a**, illustrate how the introduction of the speed limit affects the shape of all diagrams.

c) If the speed on a given day has the uniform distribution in (40, 80) km/hr, estimate the mean and variance of the flow on that day.

Problem 1.14 — Speed limit Suppose that on a freeway having no posted speed limit, the average speed of traffic, v, is related to the density, k, by a relation of the form

$v/v_f = 1\text{-}(k/k_j)^2$,

where v_f and k_j are the free-flow speed and jam density, respectively.

a) Sketch the relation between k/k_j and v/v_f, the relation between k/k_j and q/q_{max}, and the relation between v/v_f and q/q_{max} where q is the flow and q_{max} the capacity.

b) Assume that if a speed limit of $\alpha \cdot v_f$ is imposed, the above relation between v and k would be modified to

$v/v_f = min\{\alpha,\ 1\text{-}(k/k_j)^2\}$.

Using your sketches from part **a**, illustrate how the introduction of the speed limit affects the shape of the density-speed, speed-flow and density-flow relations.

c) Determine the smallest value of α that can be imposed without reducing the freeway's capacity.

Problem 1.15 — Wave speed empirical calculation The objective of this problem is to estimate the shape of the congested branch of the fundamental diagram of a freeway facility using trajectory data from NGSIM. To this end, an install the → Trajectory Explorer application. Also from the website download the trajectory images (bitmaps) for the I-80 freeway segment. Then, do the following:

1. In "line mode" estimate the wave speed for 6 obvious disturbances of your choice that travel upstream (in any lane). Report the mean and standard deviation of your measurements.
2. For this part, you will use "follow platoon mode" to compute flow-density pairs using the generalized definitions, for areas encompassing ∼10 vehicles and where traffic conditions appear to be stationary.
   (a) For lane 2, estimate several (∼10) flow-density pairs along a disturbance, trying to include a wide range of speeds (from high speeds to complete stops). Estimate a linear regression with this data; the slope o this line is the wave speed. Repeat this experiment for another ∼5 disturbances.
   (b) Test the hypothesis that the wave speed in part b1) is the same as in a).
   (c) For lane 6, estimate several flow-density pairs just downstream of the on-ramp, and also around 300 m downstream. Do you see a pattern? If yes, can you explain why?

NOTES:

1. all your measurements are automatically logged into the file "Edie.txt" on the working directory.
2. For each set of measurements, include the time-space diagram showing the areas you picked for the measurements and the corresponding flow-density diagram.

Problem 1.16 The distribution of vehicle speeds measured by a stationary observer is uniform in the interval [Vmin, Vmax]. The speed limit is V, Vmin < V < Vmax. Determine the fraction of speeding vehicles seen by a police car traveling at speed V0. Plot this fraction for -∞ <= V0 <= ∞. Discuss.

Problem 1.17 A 200 double-chair lift at a ski resort moves at 8 mph. The chair lift can hold 200 skiers, since at any point in time only 100 two-person chairs are going uphill. The distance traveled by each chair from the bottom to the top is 1/2 mile. The downhill ski run is 1-mile long. All skiers who are neither skiing on the run or riding the lift are considered to be "in queue" and they are served first in, first out. On a given afternoon, there are 500 skiers (and the lift always holds 200 of them).

a) If everybody skis downhill at 16 mph, find the number in queue and each skier's time in queue (i.e., the elapsed time from completing a run to sitting down on a lift chair).

b) Assume now that half of the 500 skiers (i.e., 250 skiers) are "experts" who always ski downhill at 24 mph and the other half are "novices" who always ski downhill at 10 mph. Determine the total number of skiers in queue and the number of those (in queue) who are "experts."

Problem 1.18 Consider a very long section of highway consisting of a travel lane and an adjacent passing lane. Traffic in the subject direction is stationary and an aerial photograph shows that 1/5 of the vehicles in the highway section are trucks and that 4/5 are cars. The cars and trucks travel at constant speeds of 50 mph and 40 mph, respectively. Cars can pass trucks with negligible delay.
For each of the following scenarios, after the cars have joined the queues, what is the average number of cars queued behind each truck? Draw trajectories illustrating each scenario.
a) Suddenly there is snow storm after which the cars can no longer pass the trucks. The cars catch-up with the trucks and follow in queue.
b) The section of highway with free passing feeds into a long section of road on which passing is impossible, but all vehicles can enter the no-passing section without delay. Within the no-passing zone, the cars catch-up with the trucks and follow in queue.
c) The passing lane is actually an empty lane normally reserved for opposing traffic. Suddenly, a long stream of vehicles appears in the opposing lane traveling at 40 mph (in the opposing direction). This opposing traffic stream prevents vehicles from passing so that cars (in the subject direction) queue behind trucks.
Problem 1.19 — Moving observer* In example 1.11 show that if the speed of the observer is time-dependent, i.e. $v_0(t)$, the formulas (1.40) do not change if v_0 in (1.40) is interpreted as the average observer speed during the measurement interval. Also show that in this case when conditions are stationary with traffic variables q, k, v then the moving observer formula (1.41) also remains valid.

2. The Kinematic Wave model

The Kinematic Wave model (KW) is the combination of the conservation law and the fundamental diagram $q = F(k)$. For a road without entrances or exists the KW model is a first-order, nonlinear and hyperbolic PDE:

$$KW : \begin{cases} \dfrac{\partial k}{\partial t} + \dfrac{\partial F(k,t,x)}{\partial x} = 0 & \text{(Conservation law)} \quad (2.1a) \\ k(t,x) = g(t,x), \qquad (t,x) \in \beta & \text{(Boundary data)} \quad (2.1b) \end{cases}$$

where $g(t,x)$ is the “data" of our problem, which is known on a boundary β.

Definition: The data is typically given as:

1. Initial values: $k(0,x) = g(x)$
2. Boundary values: $k(t,0) = g(t)$
3. initial and boundary values: a combination of the two

Definition: Homogeneous freeways When the fundamental diagram is time and location *independent*. Typically, however, inhomogeneous-freeway problems can be decomposed into a collection of homogeneous-freeway problems where

$$q = F(k)$$

that are easier to solve.

The KW model for homogeneous freeways can be written as

$$KW : \begin{cases} k_t + F'(k)k_x = 0 & \text{(Conservation law)} \quad (2.2a) \\ k(t,x) = g(t,x), \qquad (t,x) \in \beta & \text{(Boundary data)} \quad (2.2b) \end{cases}$$

where we have used subscript to denotes partial derivatives.

The term $F'(k)$ is called it the *wave speed*, and corresponds to the propagation speed of changes in density.

$F(k)$ is concave in traffic flow, and therefore $F'(k)$ is non-increasing.

2.1 The transport equation

When the wave speed is constant, $F'(k) = w$, the kinematic wave model is called the transport equation. Its solution is the building block for the KW solution.

Consider the initial value problem (IVP) for the transport equation:

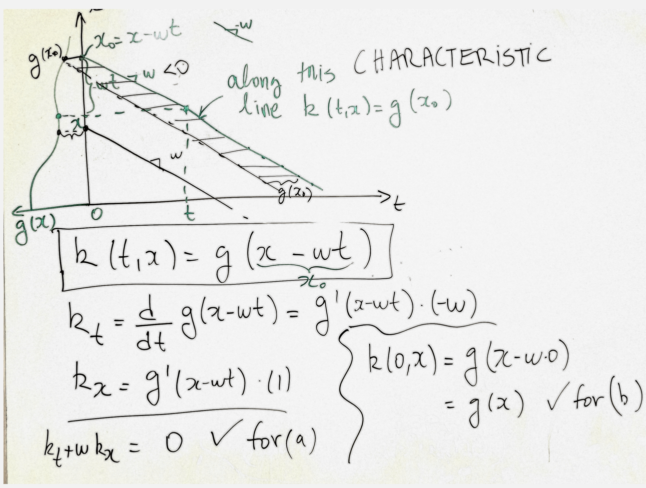
$$\begin{cases} k_t + wk_x = 0 & \text{(Transport equation)} \quad (2.3a) \\ k(0,x) = g(x), \qquad (t,x) \in \beta & \text{(Initial values)} \quad (2.3b) \end{cases}$$

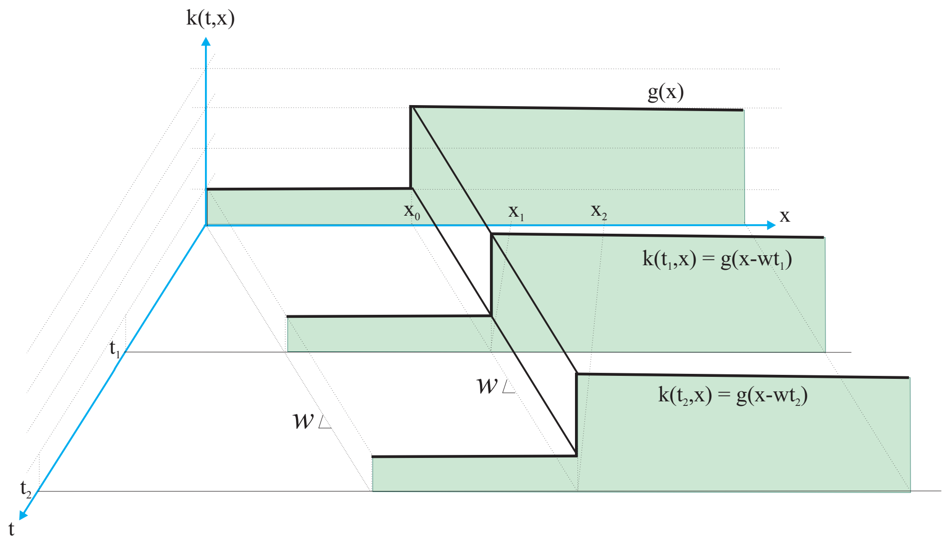
Solution of the transport equation:

$$\boxed{k(t,x) = g(x - wt)} \qquad \text{(IVP solution)} \tag{2.4}$$

This means that density is constant along the lines of slope w, which are called ***characteristics***:

$$x - wt = x_0$$





For the boundary value problem (BVP)

$$\begin{cases} k_t + w k_x = 0 & \text{(Transport equation)} \quad (2.5a) \\ k(t,0) = g(t), \qquad (t,x) \in \beta & \text{(Boundary values)} \quad (2.5b) \end{cases}$$

the solution would be

$$\boxed{k(t,x) = g(t - x/w)} \qquad \text{(BVP solution)} \qquad (2.6)$$

as can be seen in the figure. Again, the density is constant along characteristics.

2.2 The KW solution

Here we're going to use the solution of the transport equation to solve the KW model "by hand". The key idea is that when the initial conditions are piecewise constant, then in each piece the wave speed is constant and therefore the transport equation must hold, at least in a vicinity. Then, all we have to do is "stitch together" all these transport solutions.

2.2.1 Riemann problems

Riemann initial values are given by

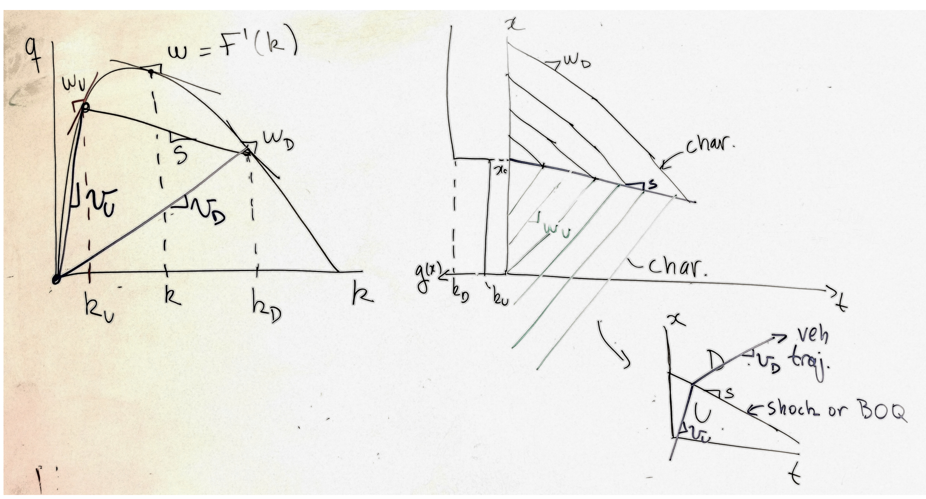
$$k(0,x) = g(x) = \begin{cases} k_U & \text{if } x < x_0, \\ k_D & \text{otherwise.} \end{cases} \tag{2.7}$$

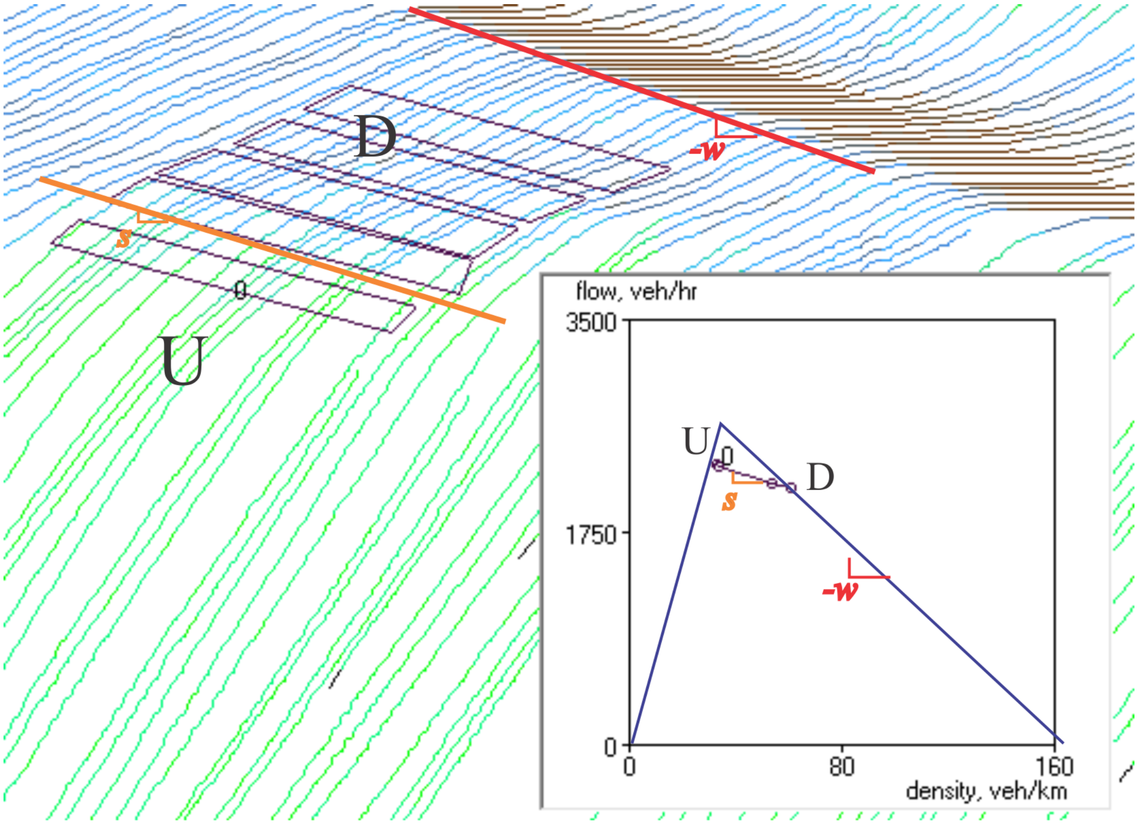
System (2.2)-(2.7) is called a "Riemann problem" in the theory of PDEs and constitutes the building-block for developing current numerical solution methods, or solutions "by hand" for the KW model. In general, Riemann problems are not trivial, but in the case of the KW model the solution is simple. There are only two cases to consider:

Case 1: $k_U < k_D \rightarrow$ **deceleration wave aka shock**

When $k_U < k_D$ the discontinuity propagates at a speed s given by the "shock condition":

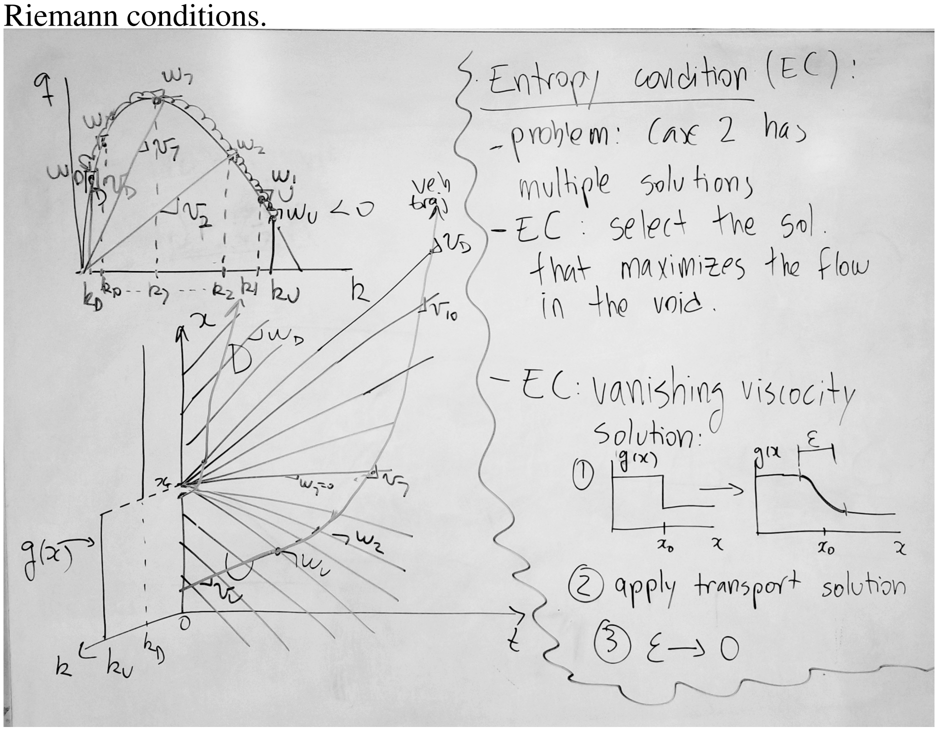
$$s = \frac{q(k_U) - q(k_D)}{k_U - k_D}, \tag{2.8}$$



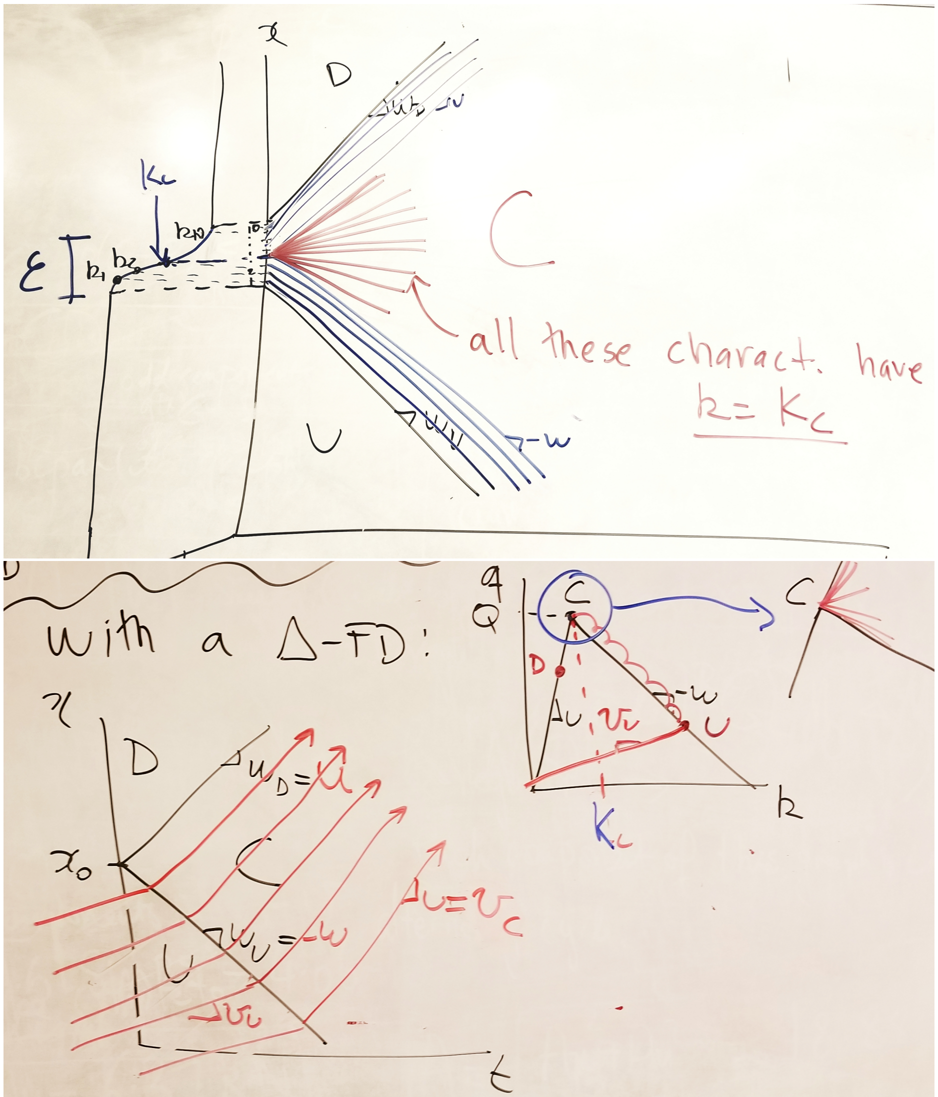


Case 2: $k_U > k_D \rightarrow$ **acceleration wave aka rarefaction fans**

When $k_U > k_D$ the solution is not unique. In fact, (2.8) is a solution among many. The *entropy condition* is used to pick a single solution: the one that maximizes the flow at each location, which can be found using the *vanishing viscosity* method. This method has the effect of “smoothing out" the jump discontinuity in the Riemann conditions.



R Notice that unlike deceleration waves, rarefaction fans have not been observed in the field. Unless the fundamental diagram is triangular, rarefaction fans produce gradual acceleration of vehicles which induces waves that spread, dissipating the discontinuity as time passes, which is unrealistic.



2.3 Examples

In all these examples we will consider a road segment with n identical lanes, each obeying a triangular fundamental diagram with free-flow speed u, wave velocity w and jam density for one lane K. The capacity of one lane is therefore given by

$$Q = \frac{wu}{w+u}K \tag{2.9}$$

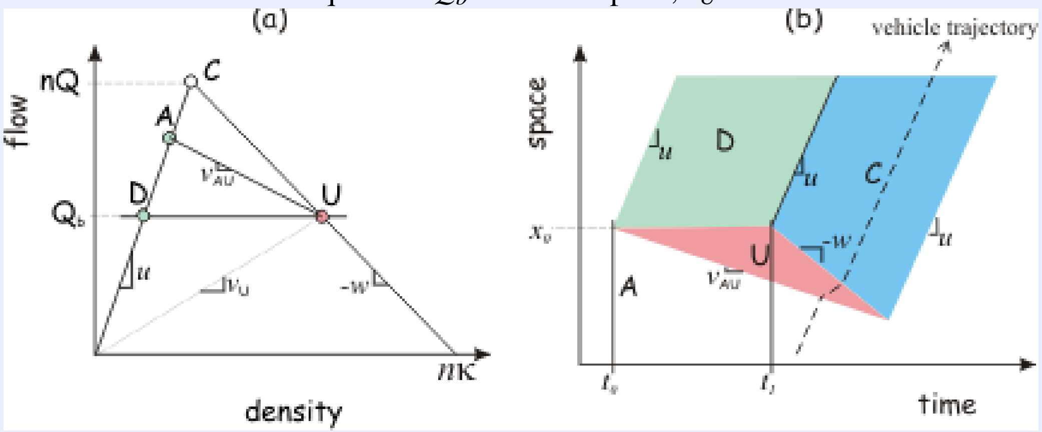
Example 2.1. — Incidents Suppose that a car breaks down in the freeway at location $x = x_0$ blocking two lanes from $t = t_0$ to $t = t_1$. Describe the impact of this incident on traffic dynamics.

One may assume that the maximum flow that can pass the bottleneck is given by the capacity of the remaining unblocked lanes:

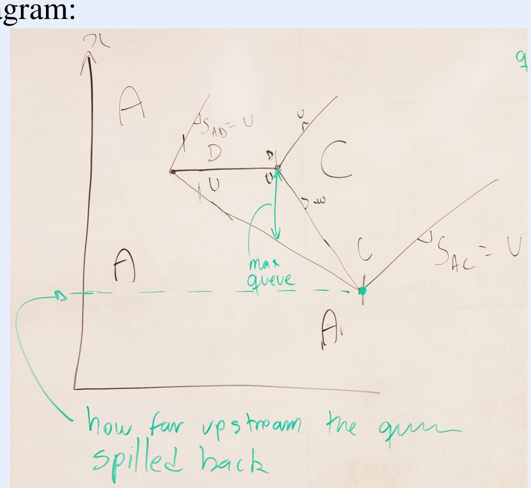
$$Q_b = (n-1)Q. \tag{2.10}$$

According to the kinematic wave theory :

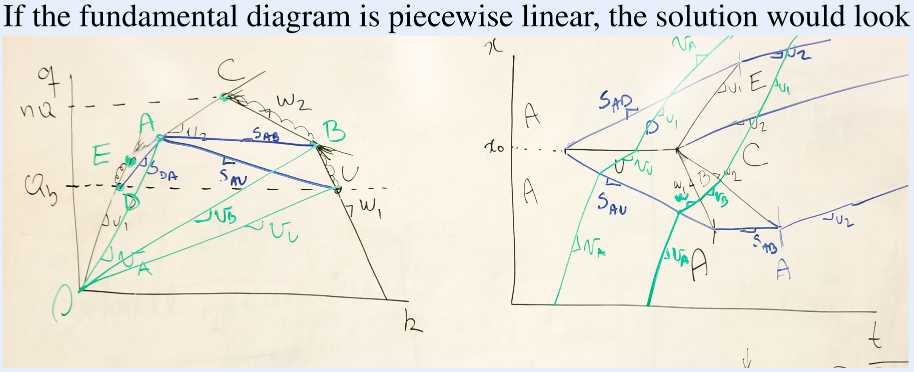
1. the queue upstream of an *active* bottleneck is in state U
2. the free-flow state downstream of an *active* bottleneck is state D
3. the flow of cars in the queue is Q_b and their speed, v_U.



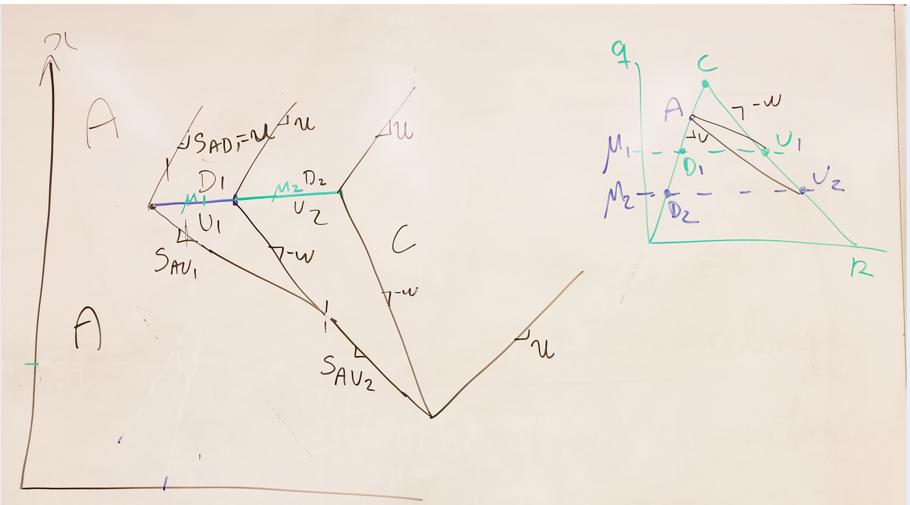
Maximum queue length and the extent of the queue can be obtained directly from the time-space diagram:



If the fundamental diagram is piecewise linear, the solution would look like this:



Finally, if the bottleneck capacity changes from μ_1 to μ_2:



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Example 2.2. — A moving bottleneck Consider a slow truck that enters the freeway, and travels from location $x = x_0$ to $x_0 + L$ at a constant speed v.

The solution to this problem is identical to the solution of the incident example, with the exception that the slope of the line connecting U and D is the speed of the moving bottleneck, v.

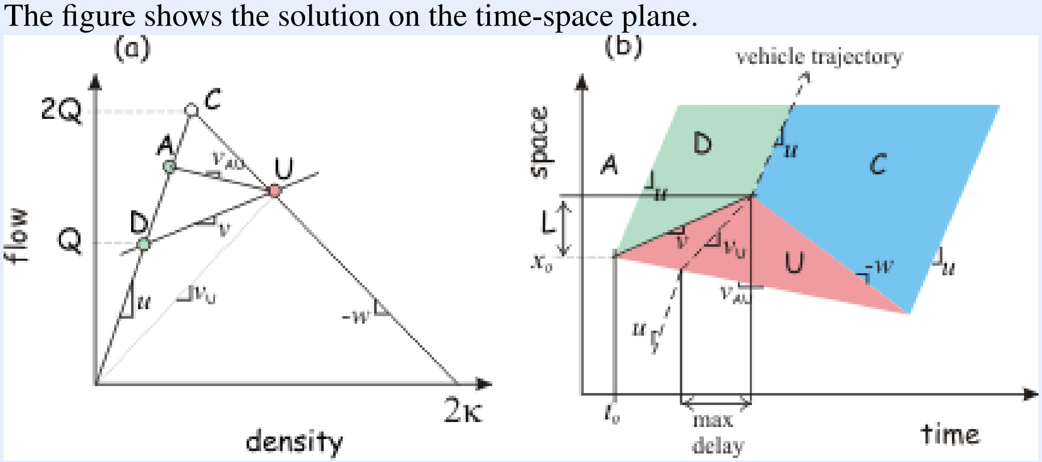
Again, one may assume that the maximum flow that can pass the truck is given by the capacity of the remaining unblocked lanes:

$$Q_D = (n-1)Q. \tag{2.11}$$

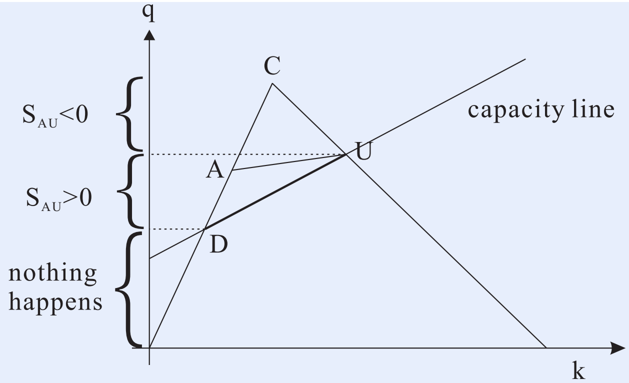
It follows that the state of the queue upstream of the moving bottleneck is given by

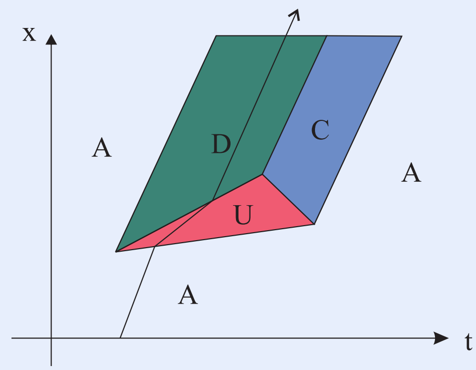
$$Q_U = Q_D + \frac{wv}{w+v}K. \tag{2.12}$$

The figure shows the solution on the time-space plane.

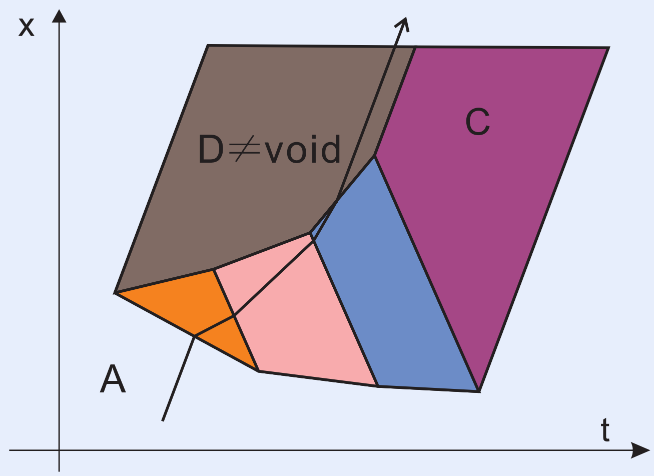


The back of the queue can travel downstream:

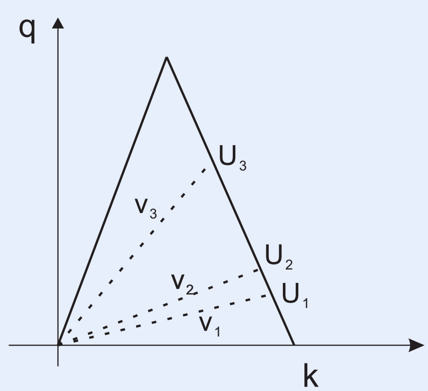


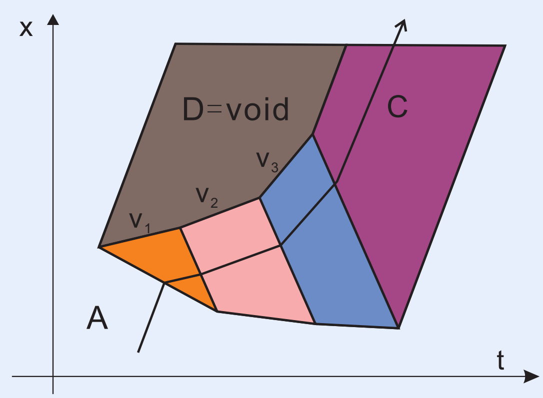


If the moving bottleneck has a piecewise linear trajectory:



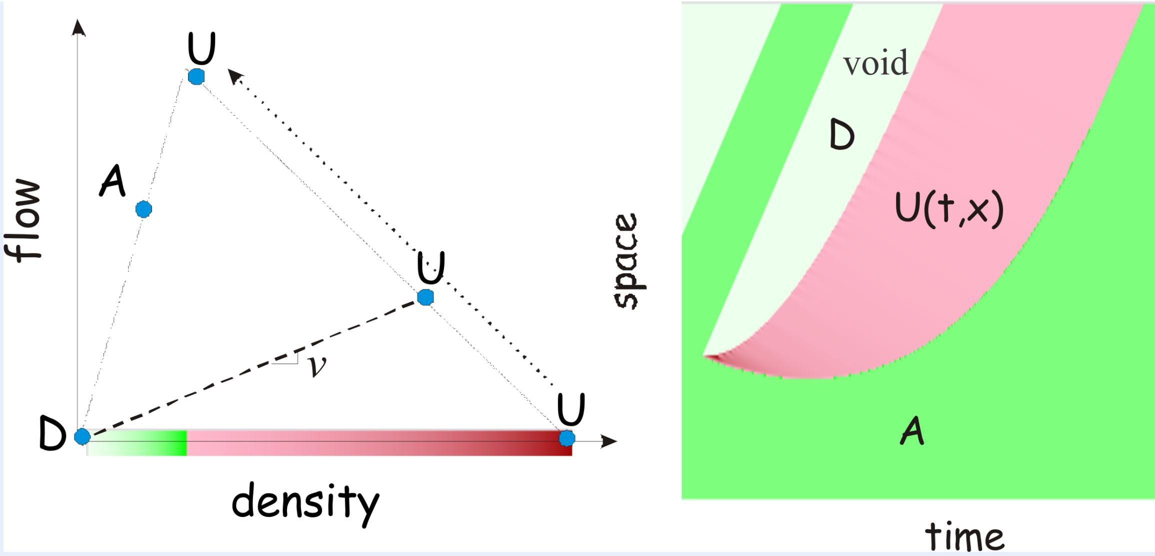
If the moving bottleneck travels **on a one-lane road**, state D becomes a void:



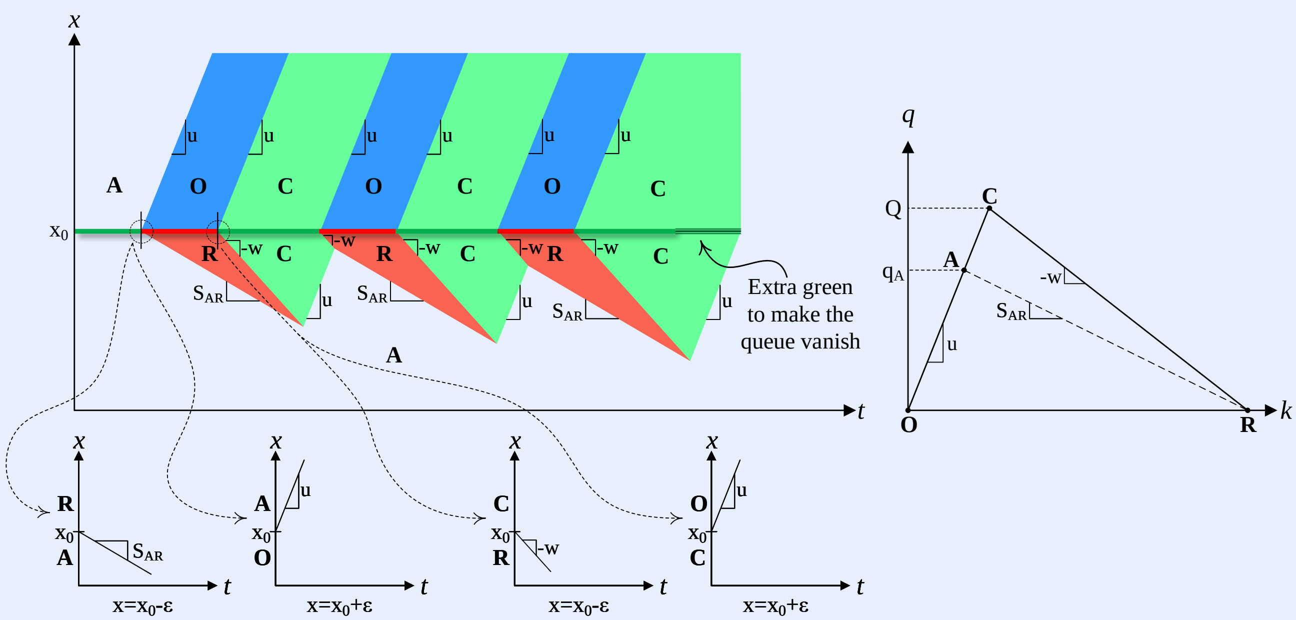


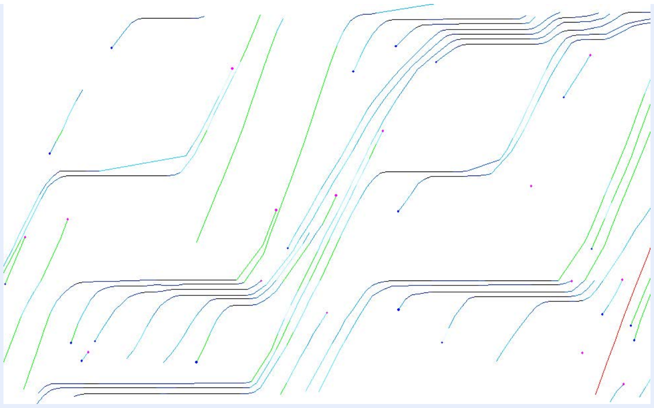
R On a one-lane road, the trajectory of the follower congestion becomes identical to the leader's but shifted along the wave speed.

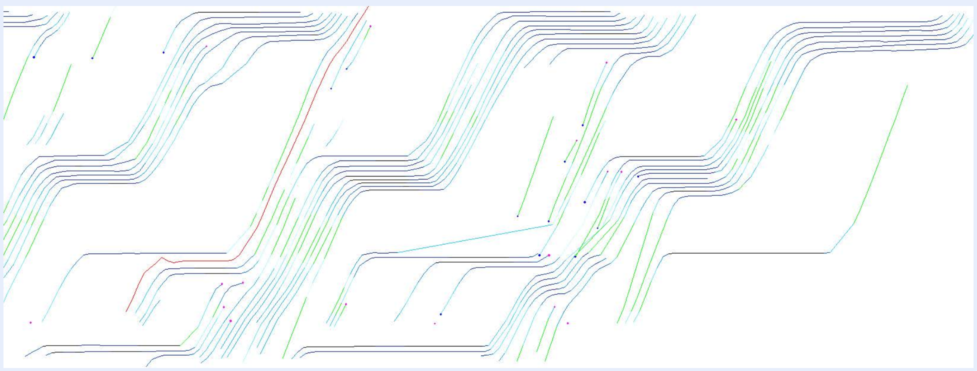
A disruptive lane change:



Example 2.3. — Traffic lights Note:
solution with Greenshields fundamental diagram: here and here
Real trajectory data from Peachtree Street in Atlanta (NGSIM):



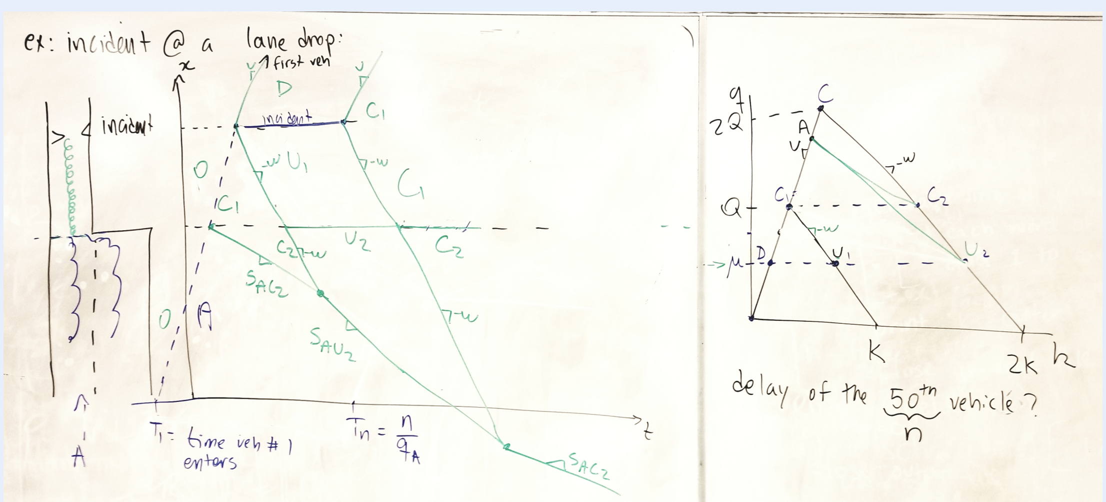




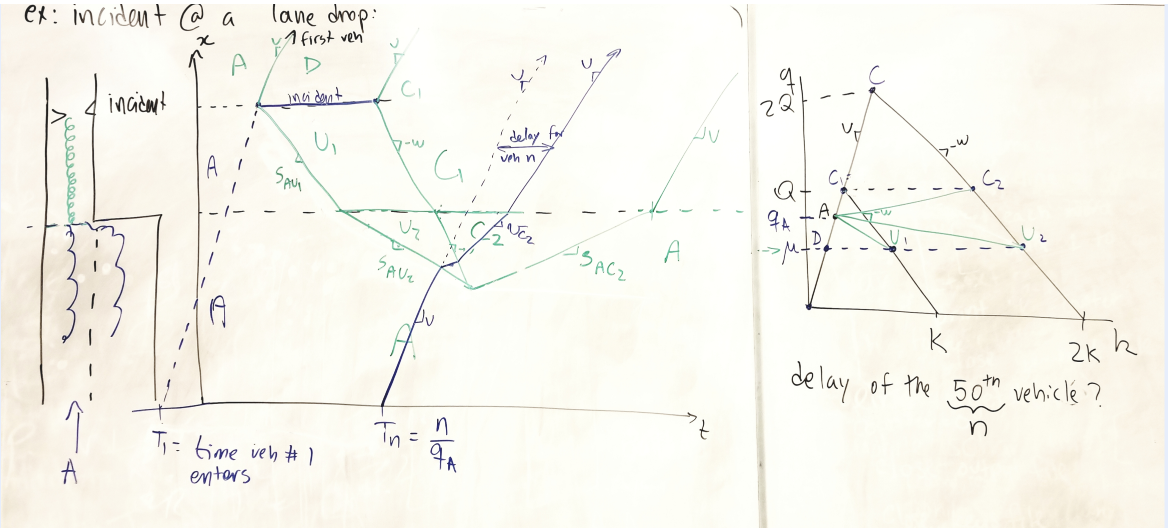
■

Example 2.4. — Incident downstream of a lane drop Consider an incident downstream of a $2 \to 1$ lane drop that reduces the capacity of the single-lane segment to μ.

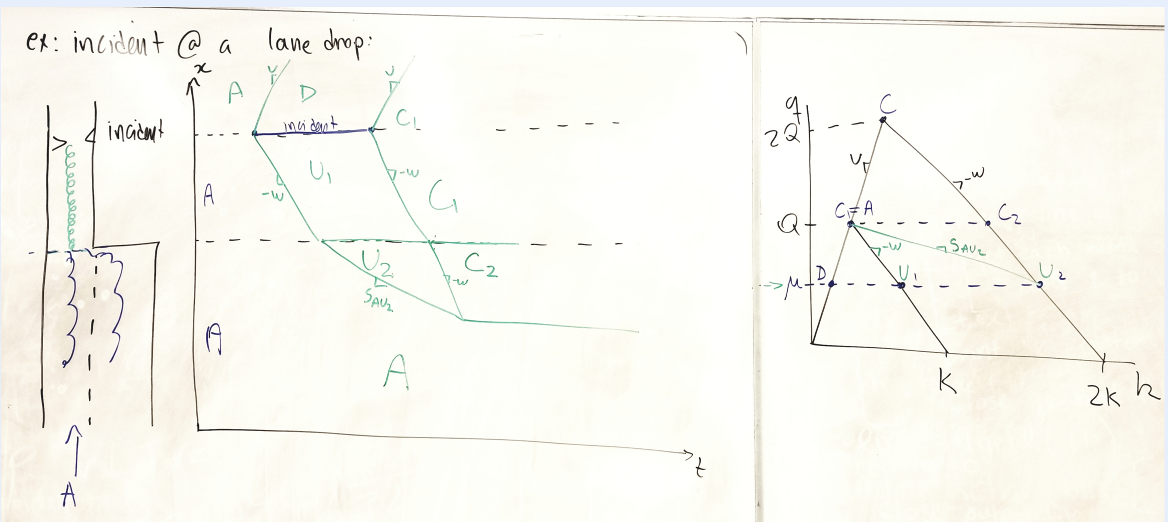
If $q_A > Q$:



If $q_A < Q$:

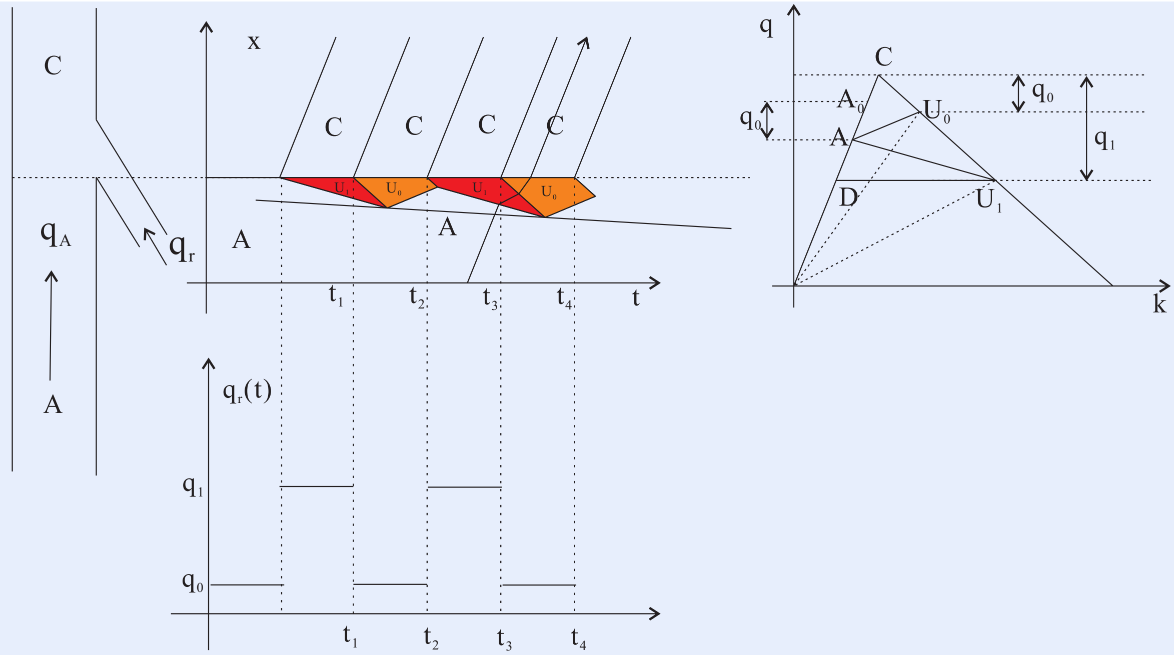


If $q_A = Q$:



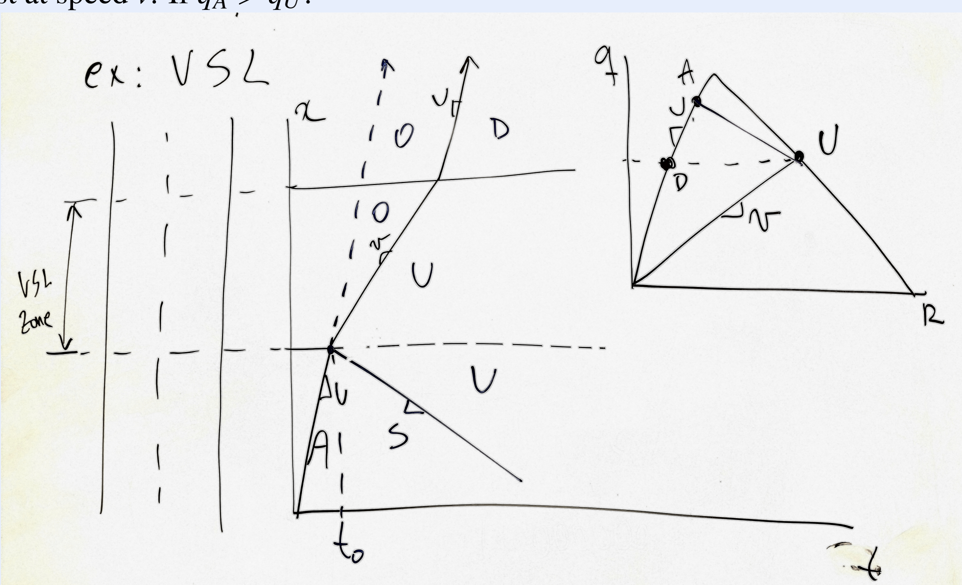
Example 2.5. — On-ramp with priority Let the on-ramp flow $q_r(t)$ be time-dependent and assume it has priority over the incoming freeway flow. As shown below, $q_r(t)$ is equal to either q_1 or q_2 such that:

$$q_A + q_1 > nQ \qquad \text{and} \qquad q_a + q_0 < nQ$$

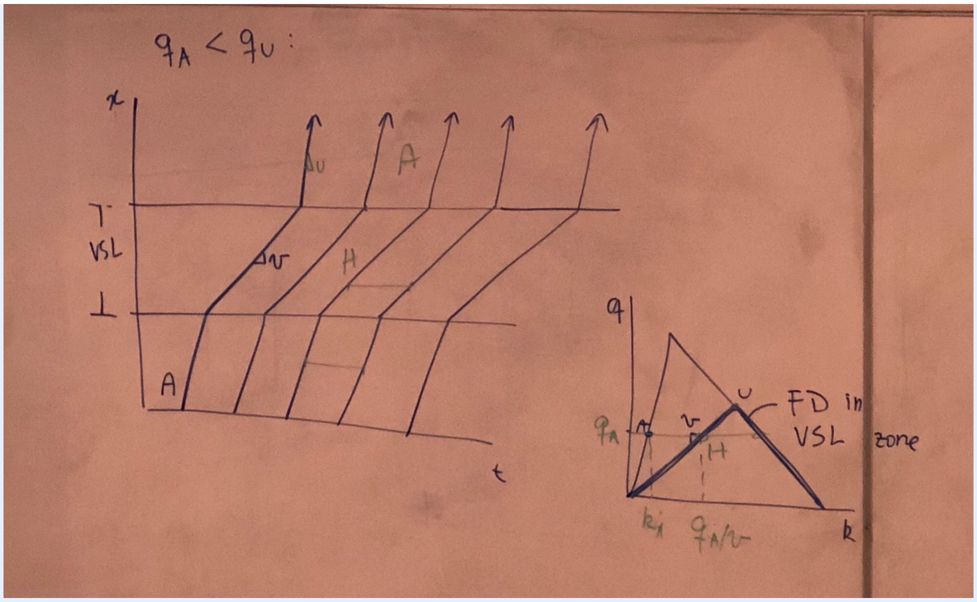


■

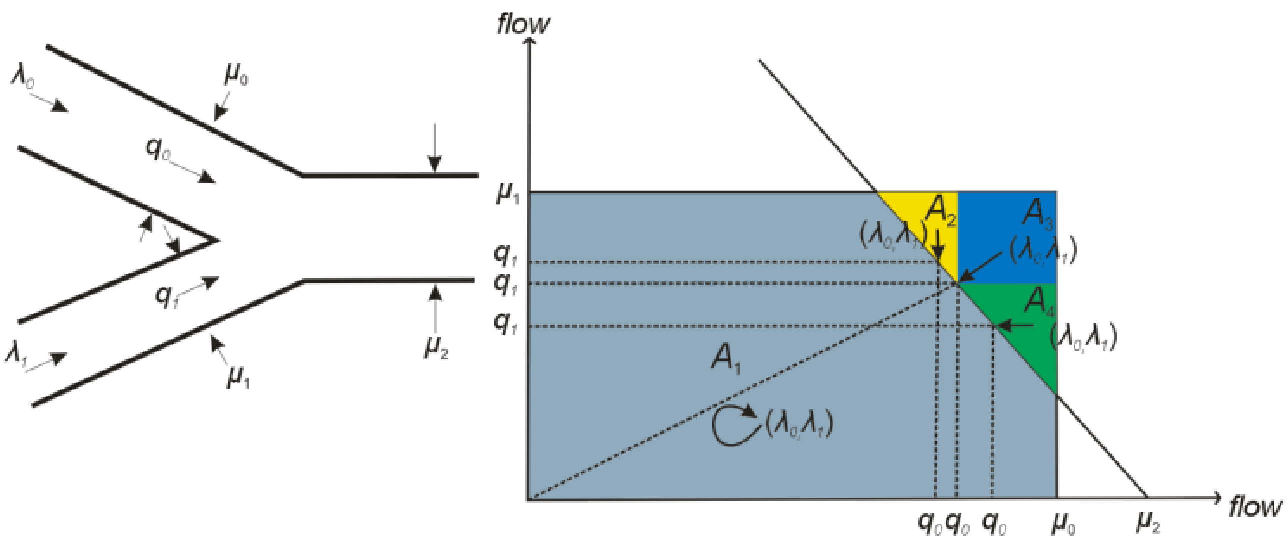
Example 2.6. — Variable Speed Limit (VSL) Inside the VSL zone of length L vehicles can drive at most at speed v. If $q_A > q_U$:

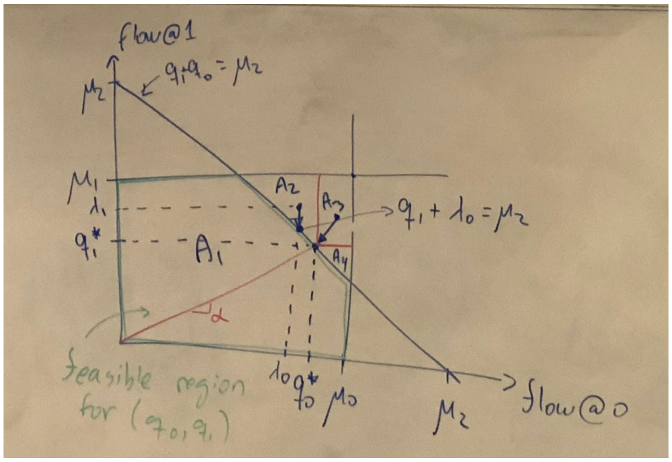


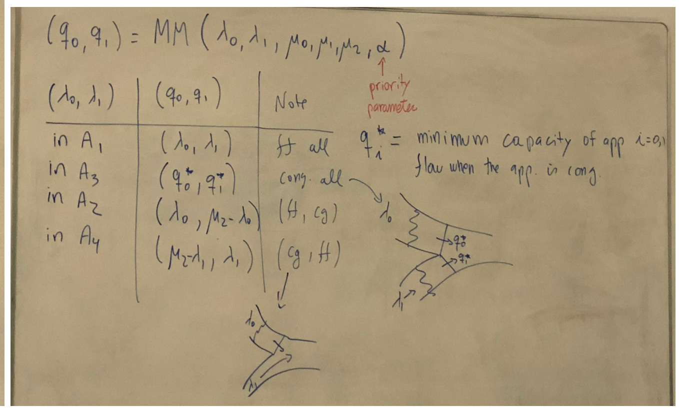
If $q_A < q_U$:



2.4 Newell-Daganzo merge model



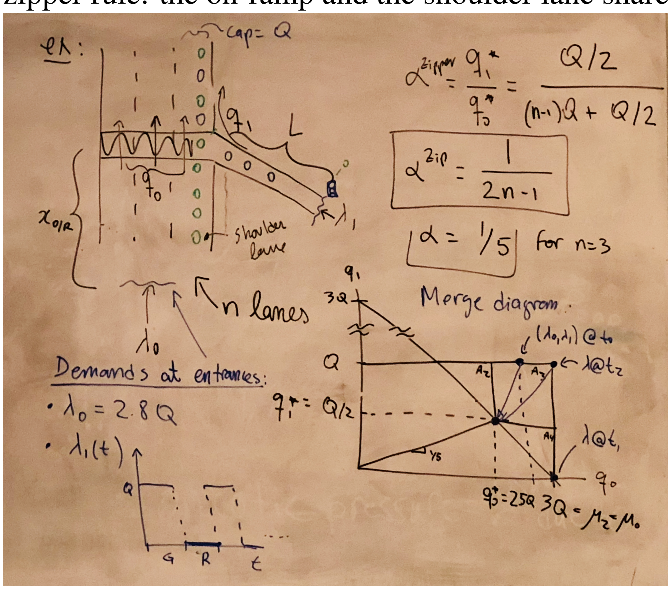


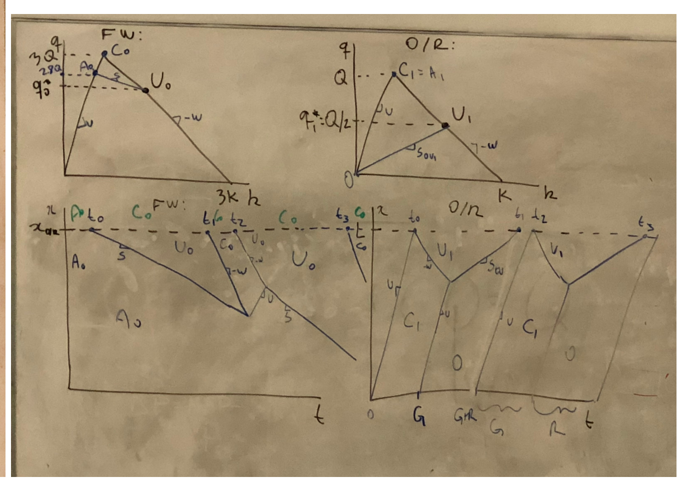


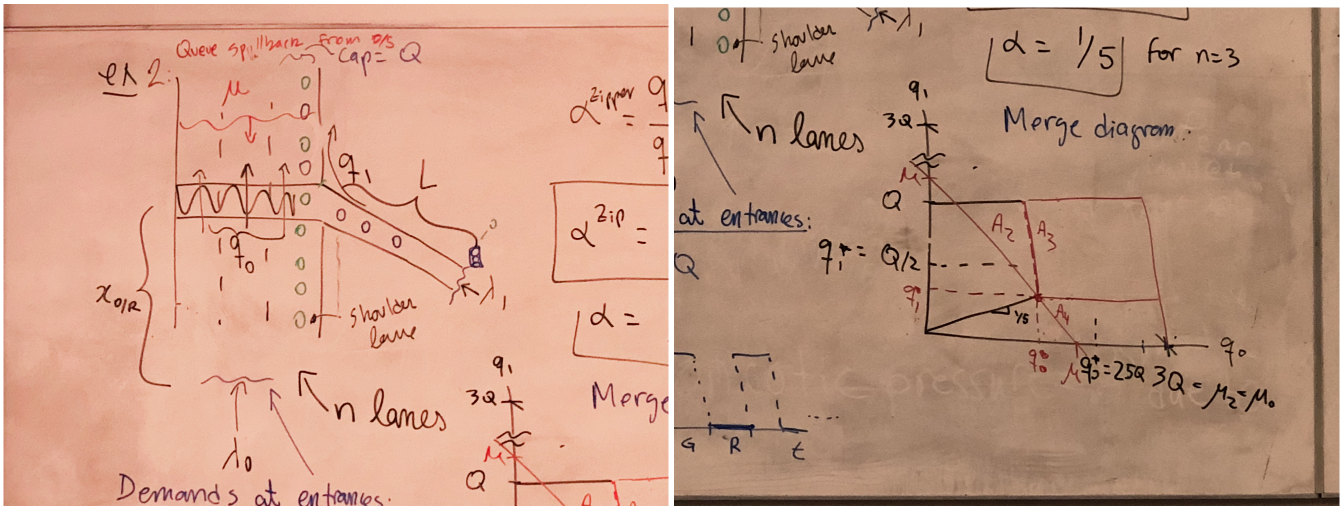
$$\boxed{\alpha = \frac{q_1^*}{q_0^*}} \qquad \text{(priority parameter)} \tag{2.13}$$

R Cassidy, M. J., Ahn, S. (2005). Driver Turn-Taking Behavior in Congested Freeway Merges. Transportation Research Record, 1934(1), 140–147. → More info...

zipper rule: the on-ramp and the shoulder lane share the capacity of the shoulder lane equally.



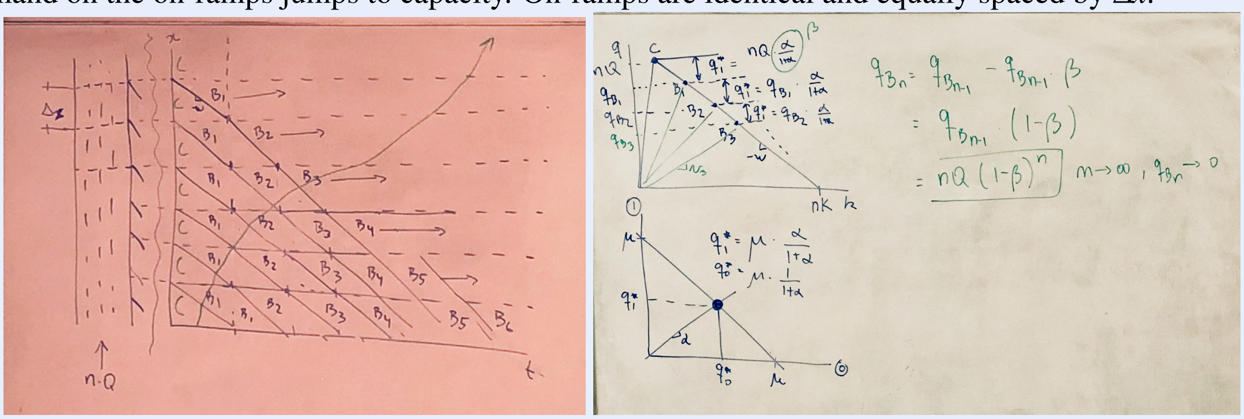




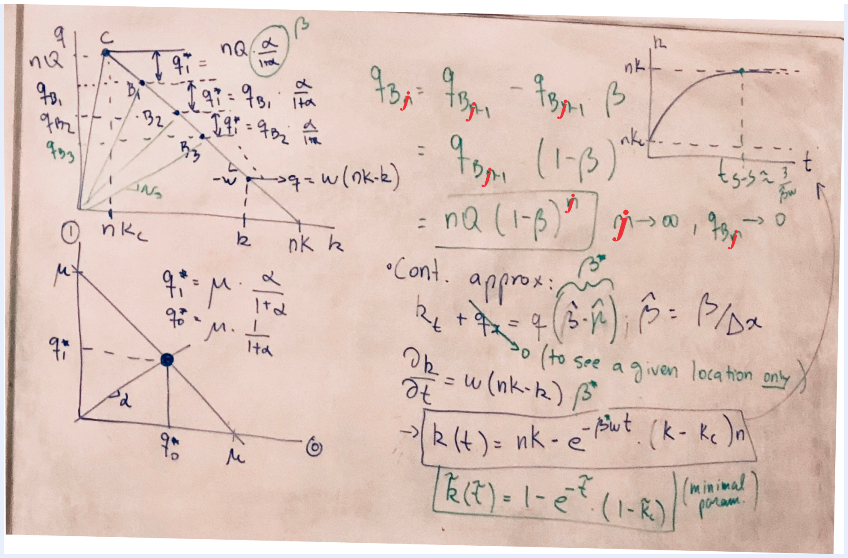
When the downstream capacity is μ:

$$q_0^* = \mu \cdot \frac{1}{1+\alpha} \qquad\qquad q_1^* = \mu \cdot \frac{\alpha}{1+\alpha} \tag{2.14}$$

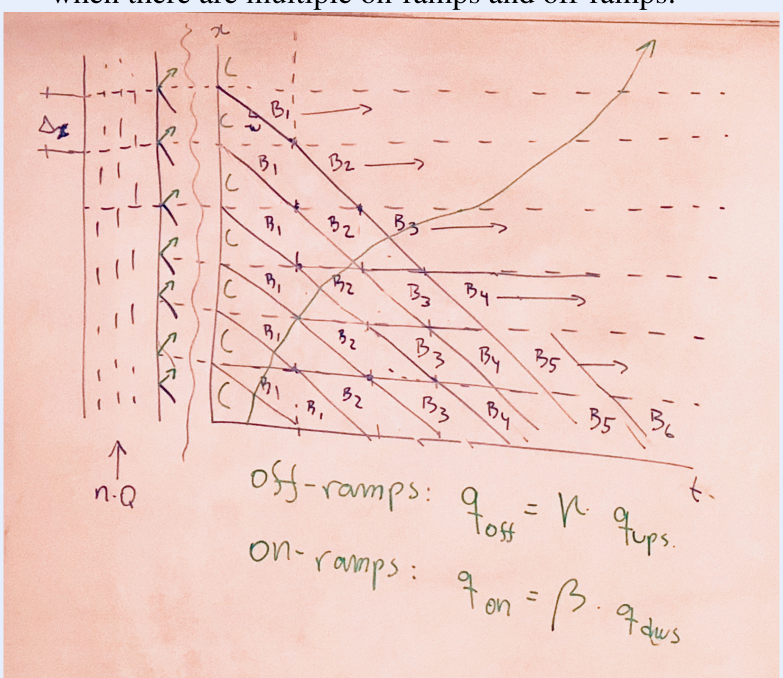
Example 2.7. — Several congested on-ramps Consider a freeway flowing a capacity when the demand on the on-ramps jumps to capacity. On-ramps are identical and equally spaced by Δx.



continuum approximation:



when there are multiple on-ramps and off-ramps:



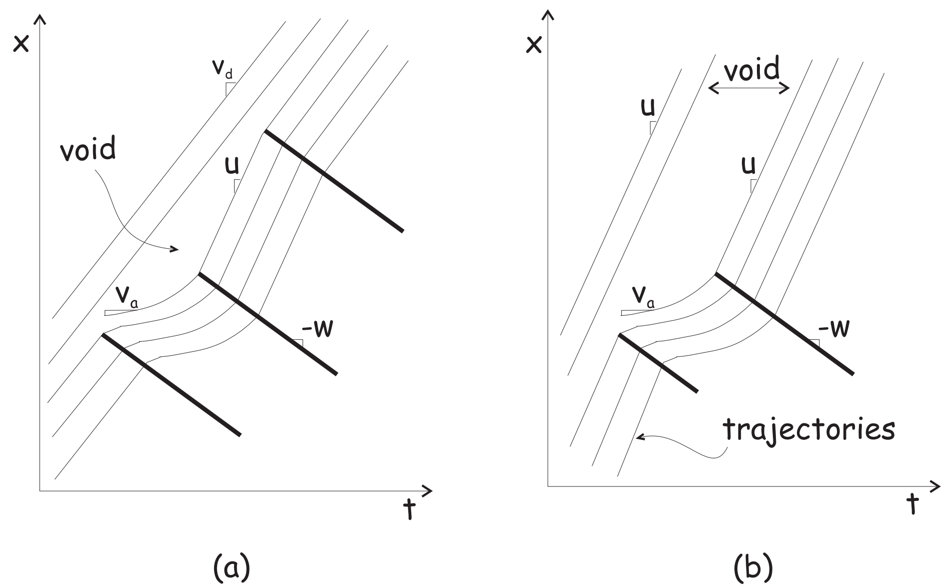
2.5 Capacity drop at merges and lane drops

PowerPoint presentation here.

See animation here.

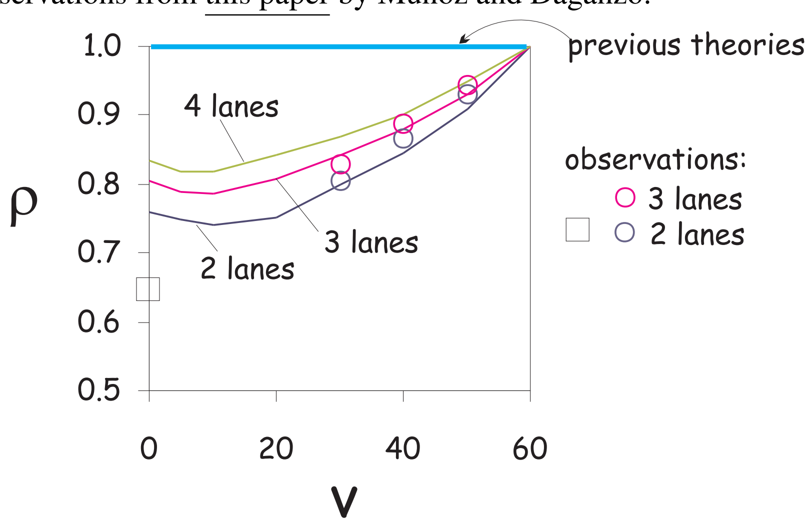
Disruptive lane changes create voids

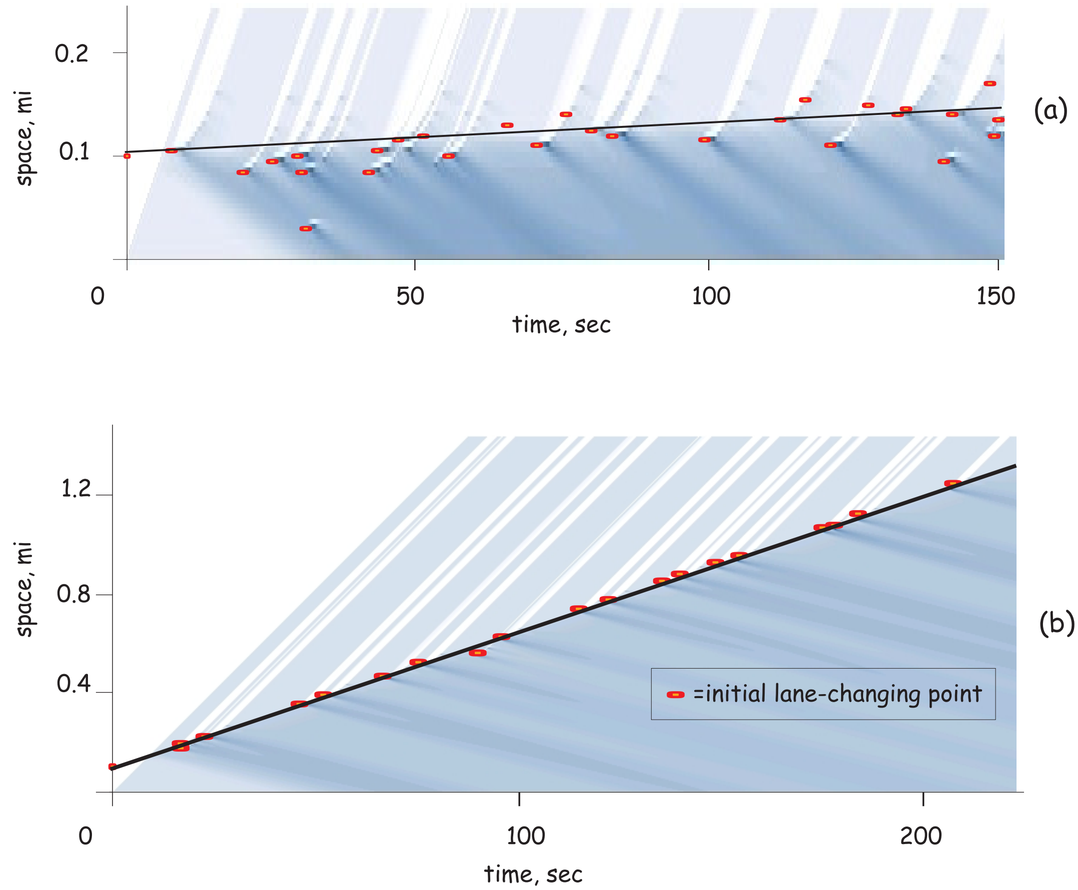
- in the origin lane if the driver slows down to find a gap, or/and
- in the destination lane because of the limited acceleration capabilities to reach the target speed



2.5.1 Along slow moving obstructions

Observations from this paper by Munoz and Daganzo:





2.6 Numerical solutions

Conservation laws are typically solved with the finite difference method. In this method, time and space are discretized in increments Δt and Δx, respectively. Then we define:

$$(t_j \doteq j\Delta t, \quad x_i \doteq i\Delta x) \qquad \text{is the numerical grid} \tag{2.15a}$$

$$k_i^j \qquad \text{is the density in cell } i \text{ during time step } j \tag{2.15b}$$

$$q_i^j \qquad \text{is the flow } \textit{into} \text{ cell } i \text{ during time step } j \tag{2.15c}$$

The discrete approximation of the conservation law becomes:

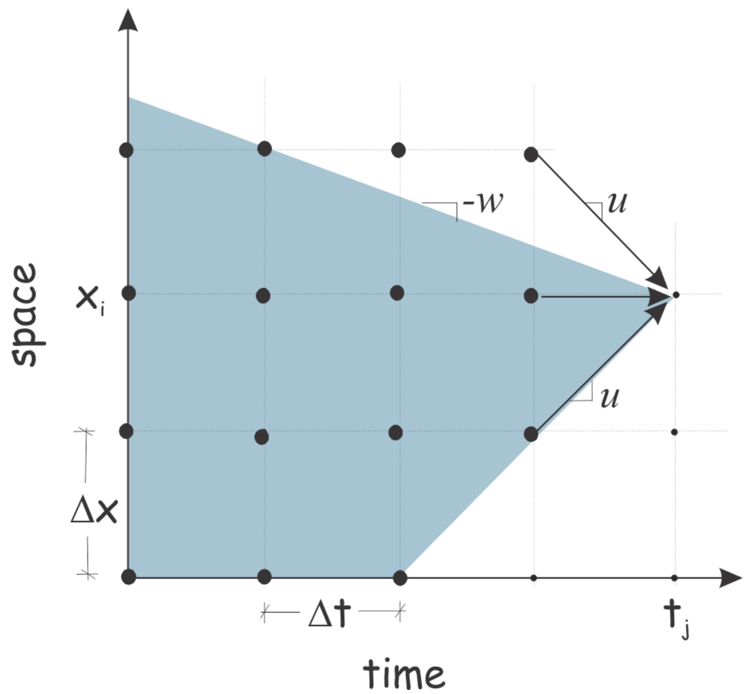
$$\frac{k_i^{j+1} - k_i^j}{\Delta t} + \frac{q_{i+1}^j - q_i^j}{\Delta x} = 0 \tag{2.16}$$

which can also be written as the update scheme:

$$\boxed{k_i := k_i + \frac{\Delta t}{\Delta x}(q_i - q_{i+1})} \tag{2.17}$$

the key question here is how to calculate the flows q_i. [1]

[1] We use the update symbol ":=" instead of $k_i^{j+1} = k_i^j - \ldots$.



Numerical domain of dependence and stencil for Godunov's method with triangular FD.

Definition: Domain of dependence The *domain of dependence of the PDE* at (j,i), namely $\mathscr{D}(j,i)$, is the (t,x) region that could affect the value of k_i^j according to the PDE; see shaded area in the figure.

Definition: Numerical domain of dependence The *numerical domain of dependence* of the scheme at (j,i) is the set of grid-points that are connected to (j,i) through a network of stencils; see bold dots in the figure.

Definition: CFL condition The Courant-Friedrich-Levy stability condition states that the numerical domain of dependence must be a subset of $\mathscr{D}(j,i)$. This is satisfied by choosing grids such that

$$\frac{\Delta x}{\Delta t} \geq \max_k \left|\frac{\partial q(k)}{\partial k}\right|, \tag{2.18}$$

which means that vehicles must spend at least one time-step inside a cell.

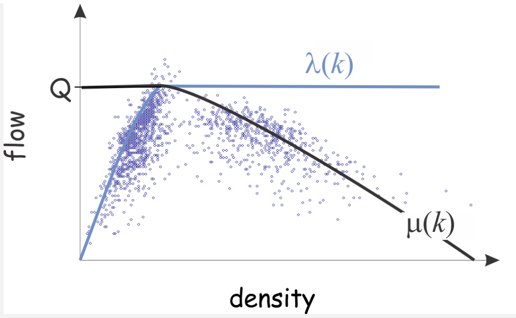
2.6.1 Godunov's method (Cell Transmission model)

When $\partial q/\partial k$ changes sign, such as in traffic flow, Godunov's method [0] is the best first-order numerical scheme. The key to Godunov's method is computing the q_i's by solving Riemann problems.

The cell transmission (CT) rule The flow $q_0(k_U,k_D)$ at $x = x_0$ of a Riemann problem (**??**) is

$$q_0(k_U,k_D) = \min\{\lambda(k_U),\mu(k_D),Q_0\}, \tag{2.19}$$

where $\lambda(k)$ and $\mu(k)$ are the sending and receiving functions (aka demand and supply functions), respectively, as in the figure, and Q_0 is the capacity of the road at $x = x_0$.



Godunov's method is then:

$$k_i := k_i + \frac{\Delta t}{\Delta x}[q_0(k_{i-1}, k_i) - q_0(k_i, k_{i+1})] \quad (2.20)$$

If the fundamental diagram is trapezoidal,

$$q_i = q_0(k^j_{i-1}, k^j_i) = \min\{Q^j_{i-1}, uk^j_{i-1}, (K - k^j_i)w\}, \quad \text{(CT rule)} \quad (2.21)$$

where Q^j_{i-1} is the capacity of the road section at (j,i). To minimize numerical errors we take:

$$\Delta x = u\Delta t \quad (2.22)$$

R Godunov's method constitutes the basis of the well known Cell Transmission (CT) model [0] assuming the triangular FD.

For a simple **spreadsheet implementation of the CTM model** click here.

Instructions: you have to download the spreadsheet and open it in your local computer and when prompted, enable macros.

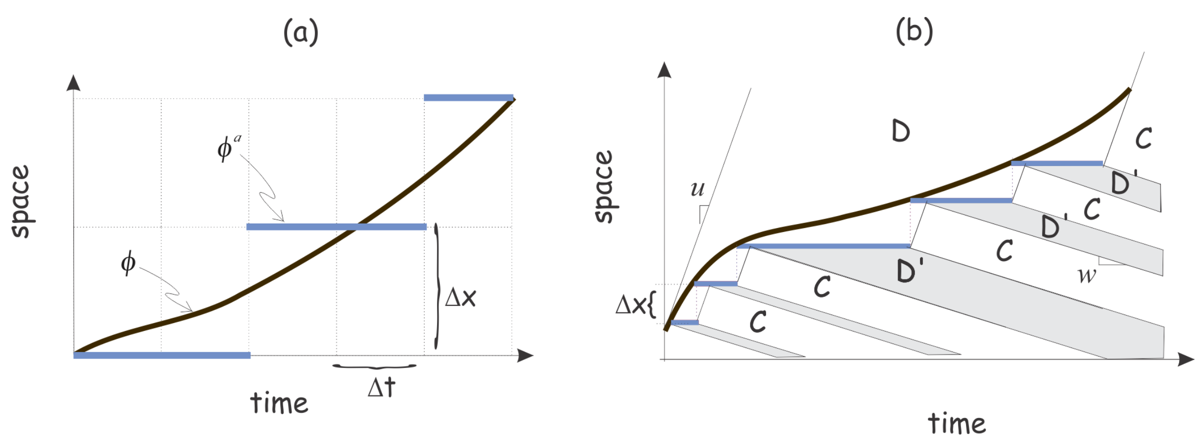
2.6.2 Discretization of a Moving Bottlenecks

We can replace the continuous trajectory of a moving bottleneck, $\phi(t)$, by an approximate step-trajectory, $\phi'(t)$, restricted to the numerical grid.

This method allows the use of "off-the-shelf" software for solving general inhomogeneous problems involving crossings.

The only disadvantage of this method is shown in part (b) of the figure: the approximate flows, densities and speeds do not converge to the exact ones, even as vehicle counts do; these quantities "flip-flop" between states C and D', the latter defined as the queued state at flow Q_D. Thus, to provide estimates one needs to average their values over multiple cells, introducing a (first-order) error.

To overcome this disadvantage, we will see in the next chapter that variational methods can be used, where any traffic problem, including moving bottlenecks, can be solved very quickly using dynamic programming. In the case of triangular FD's the method is exact.

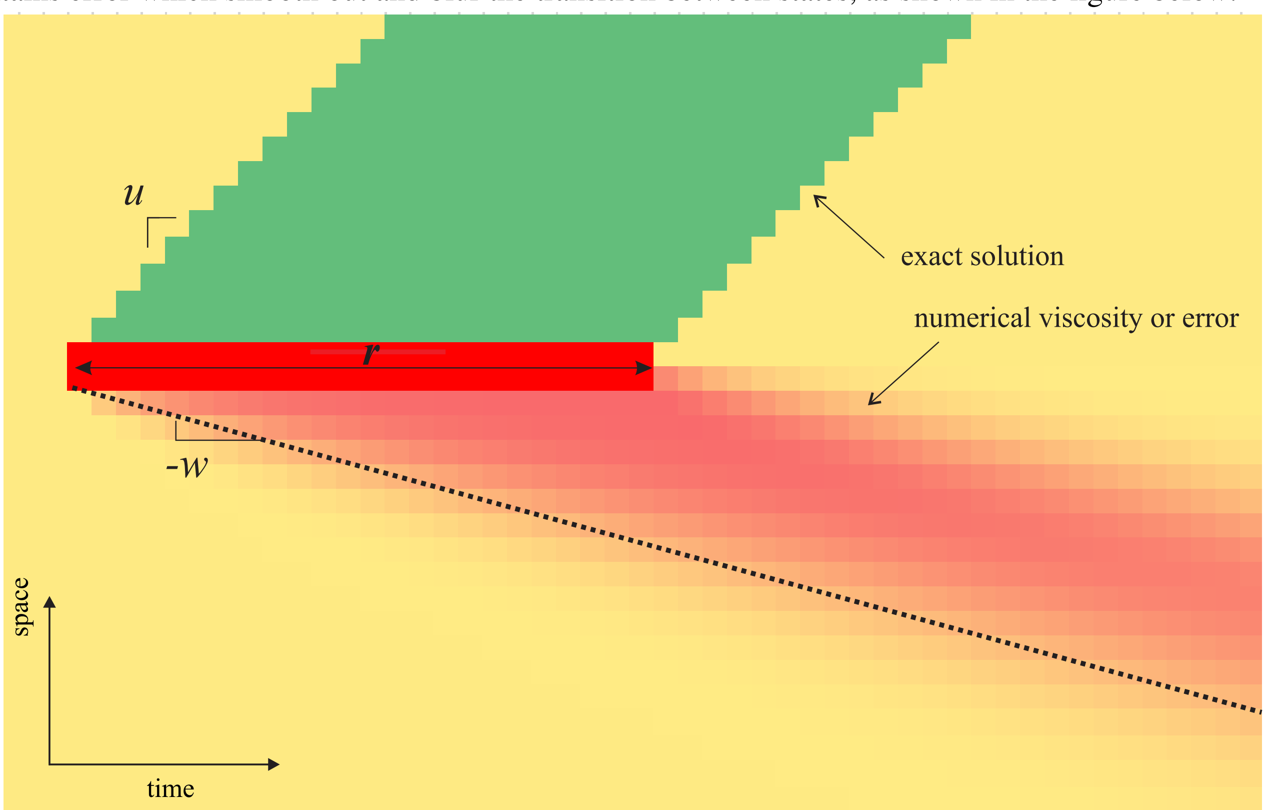


2.6.3 Numerical errors in CTM

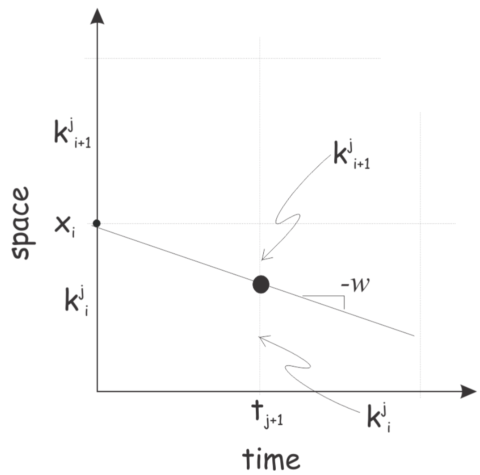
The Godunov scheme is a first-order numerical error: it converges *linearly* to the exact solution as the mesh size $\Delta t, \Delta x$ goes to zero. If Δt is reduced by a factor of two then the error is reduced by the same factor. To minimize errors it is recommended to use for the CFL condition (2.18) as an equality:

$$\Delta x = u\Delta t$$

because free-flow states propagate without error. Unless $u = w$, the propagation of congested states still contains error which smooth out and blur the transition between states, as shown in the figure below:



The figure below shows why numerical errors arise in congestion. When the initial densities are congested, the solution to the remember problem would give a wave that intersects the cell boundary at the bold dot in the figure. But because this dot is not part of the numerical grid, the update scheme (2.20) will predict a single average value for the whole cell as opposed to the two values shown in figure.



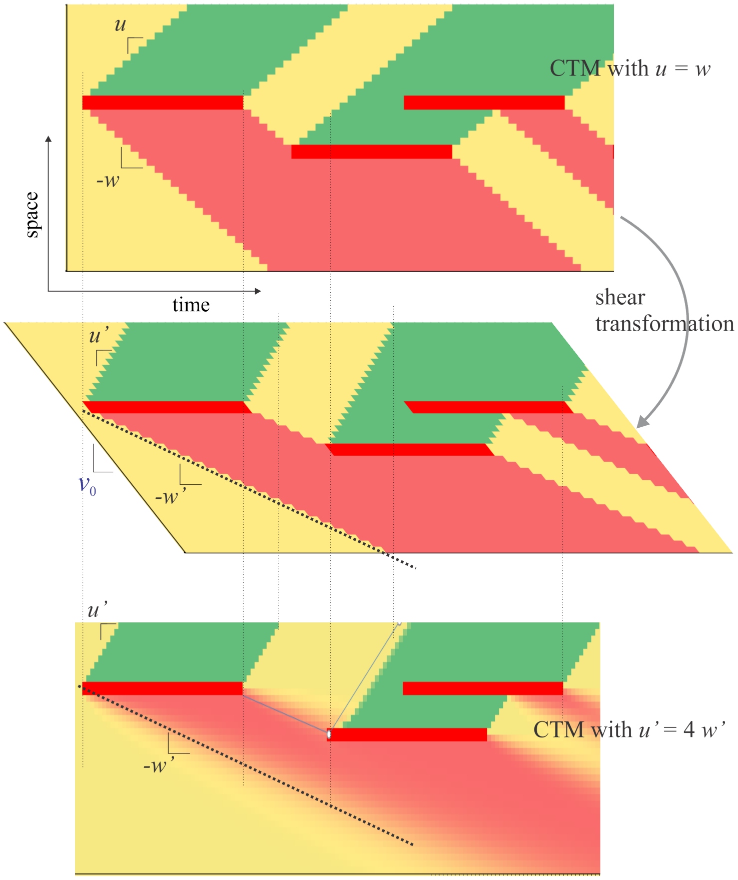
2.7 Symmetry of the kinematic wave model

PowerPoint presentation here

Shear (aka skewing) mapping from Wikipedia here

Online tool to apply sheer transformations to an image here

This chapter is based on this paper: Laval, J. A., Chilukuri, B. R. Symmetries in the Kinematic Wave Model and a parameter-free representation of traffic flow. Transportation Research Part B 89, 168 – 177, 2016.



2.8 Measures of performance

- Total time traveled in area A:

$$\Psi = \int_A k(t,x)\, dA \tag{2.23a}$$

- Total distance traveled in area A:

$$\Phi = \int_A q(t,x)\, dA \tag{2.24a}$$

- Total delay in area A:

$$\Delta = \Psi - \Psi_{ff} \tag{2.25a}$$

- Edie's average density, flow and speed in A:

$$k_A \equiv \frac{\Psi}{A}, \qquad q_A \equiv \frac{\Phi}{A}, \qquad v_A \equiv \frac{q_A}{k_A} = \frac{\Phi}{\Psi} \tag{2.26a}$$

2.9 Problems

Note Unless otherwise indicated, assume that each *lane* of the facilities in this assignment obeys a triangular fundamental diagram with $w = -20$ km/hr, K = 150 veh/km and u = 120 km/hr for freeways and u = 80 km/hr for arterials. Also, let Q and K_c be the resulting capacity and critical density, respectively. Make all the necessary assumptions, if needed.

Problem 2.1 — 2 traffic lights* Two consecutive intersections that are 100 ft. apart on a one-way street are controlled by identically set, pre-timed traffic signals. Their cycle is one minute and the effective green phase for through traffic is 30 secs. Assuming no turning movements, determine:

a) the t-x diagram, capacity of the system and average travel time per vehicle if the offset is 10 seconds.
b) what values of the offset ensure that there is no spill back (and therefore no capacity is lost) ? it is suggested that you use the CTM implementation seeing class to answer this question.

Problem 2.2 — Wide intersection

Consider a two-lane approach to a signalized intersection that turns to three lanes L1 m upstream of the intersection. Assume that there are no turning movements, that downstream of the intersection there are three lanes, that $u = 40$ km/hr and that the total demand willing to cross the intersection is $q_A = 1.2Q$. The traffic light has a red time of 30 seconds, and the green time is long enough so that all vehicles in the queue are able to cross the intersection before the light turns red again.

a) determine the maximum length of the queue and the maximum distance traveled by the queue upstream of the traffic signal,
b) determine the minimum green time so that the last vehicle in the queue is able to cross the intersection without having to stop again.

Now assume there is a lane-drop L2 m downstream of the intersection. For simplicity, consider that the length of the intersection can be neglected (ie, L0 = 0).

1. determine the minimum length L2 so that the queue forming at the lane-drop does not spill back to the intersection,
2. show that if the condition in part d) is fulfilled then the first vehicle crossing the intersection when the light turns green will find no queues at the lane-drop.

Problem 2.3 — Wide intersection design

Consider the signalized intersection problem seen in class where a two-lane road widens to three lanes a distance $l_1$upstream of the intersection and then goes back to two lanes a distance l_2 downstream of the intersection. The signal is pre-timed with constant red and green times, r and g. You are asked to design the intersection in order to maximize capacity; i.e., what are the values of $l_1$and l_2 as a function of signal settings and the (triangular) fundamental diagram parameters.

Problem 2.4 — Wide intersection capacity* A one-lane approach to a signalized intersection widens to two lanes 1/15th of a mile (352 ft) upstream of the intersection; i.e., the approach has two lanes at the intersection, but only one lane further upstream. The flow-density diagram (i.e., relation) for the one-lane section is linear between the following k (veh/mile), q (vph) "break points":

$$0,0 \quad 18.75,900 \quad 75,1800 \quad 150,0$$

a)Assuming that the diagram for the two-lane section exhibits the same speed for twice the density, draw both diagrams on the same graph. (Use only the "top half" of a sheet of paper).

b)What is the capacity of the intersection approach if the traffic signal has a 60-second cycle and a 30-second effective green? To solve part b, you should draw a time-space diagram on the bottom half of the sheet of paper you used for part a. For simplicity, you may assume that vehicles accelerate and decelerate instantaneously.

Problem 2.5 — Single merge* Traffic flows at 2.5Q on a long three-lane freeway segment upstream of a single-lane on-ramp flowing at 0.4Q. These demands remain constant and you may assume the zipper priority merging. After 30 minutes of operations under these conditions, a downstream queue spills back to the merge from a downstream incident that lasts for 20 minutes; the flow in that queue is 2Q.

1. draw time-space diagrams for the on-ramp and the freeway showing the different traffic states that arise until the congested episode vanishes. Also include the relevant fundamental diagrams and merge diagram. Clearly show all traffic states in all diagrams.
2. Estimate the total travel time for freeway and on-ramp users, and compare the total aggregated travel time with a situation with similar total demand but no merge. Comment.
3. Qualitatively, how would your answer in part b) change had you considered capacity drop.

Problem 2.6 — 5 merges Consider a 2.5 km two-lane freeway segments with five identical and evenly-spaced on-ramps located at x = 0.5, 1, 1.5, 2 and 2.5 km, and measuring 500 m each. The demand on the freeway is 2,000 vph and on each on-ramp 1,000 vph. Entering vehicles have the same priority as vehicles on the freeway shoulder lane (ie, a "zipper" rule). At t=0 the system is empty and starts flowing simultaneously at its 6 entry points

a. Sketch a time-space diagram showing the different traffic states that arise the first 15 minutes
b. At what time will congestion reach the freeway entrance at x = 0.
c. What is the queue length at t=15 on each on-ramp?

Assume that each lane of the facilities obeys a triangular fundamental diagram with w = -20 km/hr, Kmax = 150 veh/km and Vmax = 100 km/hr. Also, let Q and kc be the resulting capacity and critical density, respectively. Make all the necessary assumptions, if needed.

Problem 2.7 — TBA

Problem 2.8 — Moving bottleneck at lane drop* Traffic flows at 2,000 veh/hr in the direction of increasing x on a one-directional road. At location x=0 the road narrows from 2 to 1 lanes. We assume that a flow-density curve, triangular in shape, defines the possible traffic states at all points in time-space. The following parameters apply to the 1 lane section: Free flow speed u = 80 km/hr, jam density K = 75 veh/km, and optimum density ko = 25 veh/km.

(a) Plot the flow vs. density curves for the 1 and 2 lane road sections on the same diagram.

(b) Sketch the speed vs. density curve for the 1 lane section. Show clearly where the curve is linear and where it is not.

(c) If a truck located at x = 1 km suddenly slows to v = 40 km/hr (at t = 0), describe in words (one or two sentences) what you think will happen behind it.

(d) Draw a time-space diagram with the relevant interfaces between traffic states and a couple of vehicle trajectories. Find when the effect of the speed reduction is felt at x = 0.

(e) When is it felt at x = -1 km?

(f) If the truck resumes a speed of 80 km/hr at x = 2 km, complete the time space diagram. Include a few more vehicle trajectories.

(g) What is the delay caused by the disturbance to the 100th vehicle behind the truck?

Problem 2.9 — Deviations from Newell's model The objective of this problem is to estimate how well does Newell's car-following model compare with real trajectory data, and to better understand driver behavior. To this end, go to this website http://trafficlab.ce.gatech.edu/tools.html an install the Trajectory Explorer (v2) application. Also from the website download the trajectory images (bitmaps) for the I-80 or US-101 freeway segment. Then, do the following:

1. In "square mode" select a region containing the trajectory you want to analyze. These trajectories should exhibit changes in speed. Right-click and select "ToVector". This will generate the file "traj.sgv" in your working directory. Note that only the trajectories crossing the bottom edge of the square will be included.
2. Open the generated file in your favorite vector graphics editor (e.g., CorelDraw, Inkscape at http://www.inkscape.org/, etc)

3. Ungroup the trajectories. Click on the first trajectory and slide it along the wave speed trying to match the follower's trajectory.
4. Repeat this for all trajectories in the platoon, and process approximately 6 platoons.
5. Discuss your findings.

Problem 2.10 — Ramp metering flushing Consider a four-lane freeway and a single metered on-ramp. The metering rate is determined such that the total freeway flow downstream of the merge does not exceed the freeway capacity $\mu = 4Q$, if possible. To prevent impacts to the surface streets, a "flushing" strategy is implemented: when the on-ramp queue reaches its entrance located 300 m upstream of the merge, the ramp metering device is turned off altogether, only until the on-ramp queue clears. Assume a "zipper" rule priority and that from time t = 0 to time t = 1 h, traffic arrives at the freeway at a rate $q_1 = 3.5Q$ and at $q_{r,1} = .8Q$ at the on-ramp; after t = 1, vehicles arrive at rates $q_2 = 2Q$ and $q_{r,2} = Q/4$, respectively. Determine:

a) the total travel time without the flushing strategy (freeway users and on-ramp users separately)
b) the total travel time with the flushing strategy (freeway users and on-ramp users separately)
c) total system travel time without metering altogether
d) discuss your results

Problem 2.11 — Ramp metering flushing, cont'd If there is an alternative route that bypasses the merge in the problem above, which takes an extra 10 minutes, recalculate part a) above assuming system optimum and user optimum equilibrium, using:

1. predictive travel time
2. reactive travel time
3. compare and comment

Problem 2.12 Suppose you are stopped at a traffic light and there are 10 vehicles in front of you in the same lane.

a) How long after the light turns green will you be able to i) start moving? and ii) cross the intersection?
b) Suppose that vehicle number 5 is a heavy truck that can accelerate at a constant acceleration of $0.5m/s^2$. How would your answers in a) change?

Problem 2.13 — Intersection capacity with autonomous vehicles To increase the capacity of signalized intersections one could have all vehicles in the queue accelerate simultaneously as soon as the light turns green, instead of waiting for the wave to come. Comment on the potential capacity increases relative to the safety and energy costs at different AV penetration rates.

Problem 2.14 — Capacity drop approximation* Consider an isolated on-ramp located at x = 0 on a two-lane freeway. In a simple attempt to model the capacity drop phenomenon, suppose that when both the freeway and the on-ramp are congested a certain proportion of incoming on-ramp traffic will create disruptive lane changes (DLCs) in the freeway shoulder lane. That is, we only consider the DLCs coming from the on-ramp into the freeway shoulder lane, ignoring the DLCs that may take place between the shoulder and the median lane in the freeway. Once in the shoulder lane, DLCs accelerate from an initial speed v_0 at a constant acceleration a=2 m/s^2. The proportion of DLCs is such that they take place as soon as the queue generated by the previous one has cleared at x = 0.

a) Derive formulas in terms of the variables given above for:
   (i) the time and the distance it takes a single DLC to attain maximum speed,
   (ii) the time it takes, τ, for the queue generated by a DLC to clear at x = 0.
   (iii) the capacity of the bottleneck, and the percent drop in capacity compared to a situation with no DLCs (ie, 2*Q).
   (iv) comment on the influence of the acceleration rate a on your answers in (iii).
b) Assume that v_0 is given by the speed in the on-ramp queue predicted by the kinematic wave model assuming the "zipper rule" (and no DLCs); ie, v_0 is the speed corresponding to the minimum

capacity q_1^* in the merge model seen in class. What would be the percent drop in capacity with such a value for v_0?

c) What value of v_0 would predict a capacity drop of 5%?

Problem 2.15 — Offramp bottleneck Traffic leaving a major event enters a 3-lane freeway at a major on-ramp located 2 miles downstream of the nearest off-ramp. The event traffic reduces the available capacity to the through vehicles as follows:

Qmax = 100 veh/min $t < 0$ min
= 60 veh/min $0 < t < 10$ min
= 80 veh/min $10 < t < 20$ min
= 100 veh/min $t > 20$ min

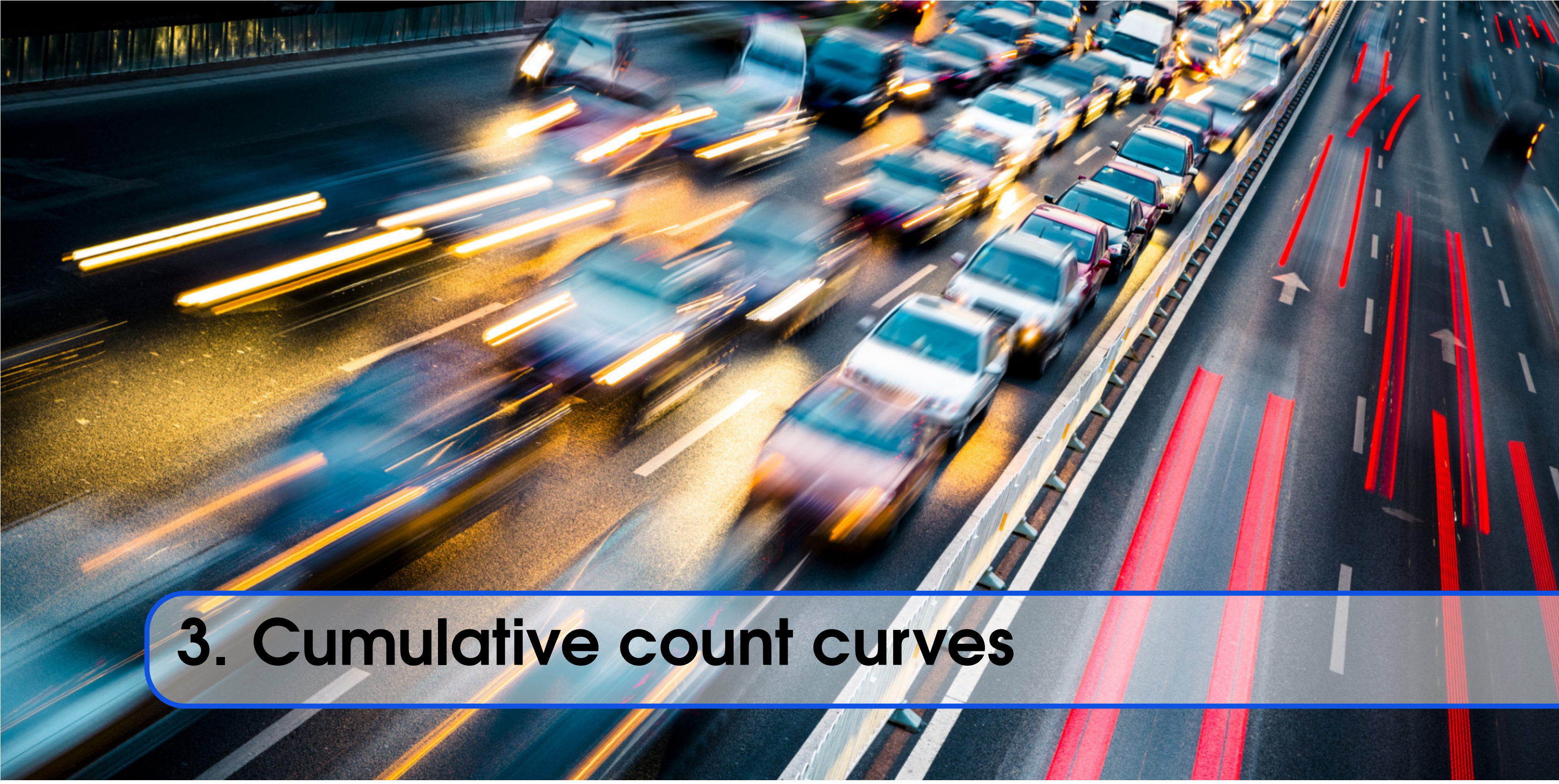
(a) If the approaching freeway traffic is time independent with q=100 veh/min and 15% of the traffic is bound for the off-ramp, determine whether or not the freeway queue will back up to the off-ramp.

(b) Explain what would happen after the spillover.

(c) Repeat the exercise assuming that traffic is metered on the on-ramp so that the Qmax will never dip below 76 veh/min. (i.e. so that the Qmax values would be 100, 76, 76 and 100 if we assume that the same number of vehicles enter).

Problem 2.16 — capacity of uphill grades Consider a 3-lane roadway with light vehicles and two types of trucks: heavy trucks that travel at a speed v1 inside the uphill of length L, and light trucks that travel at v2, inside this segment (Vmax>v2>v1). Elsewhere both trucks travel at Vmax. Let r1 and r2 be the proportion of heavy and light trucks in the traffic stream, respectively.

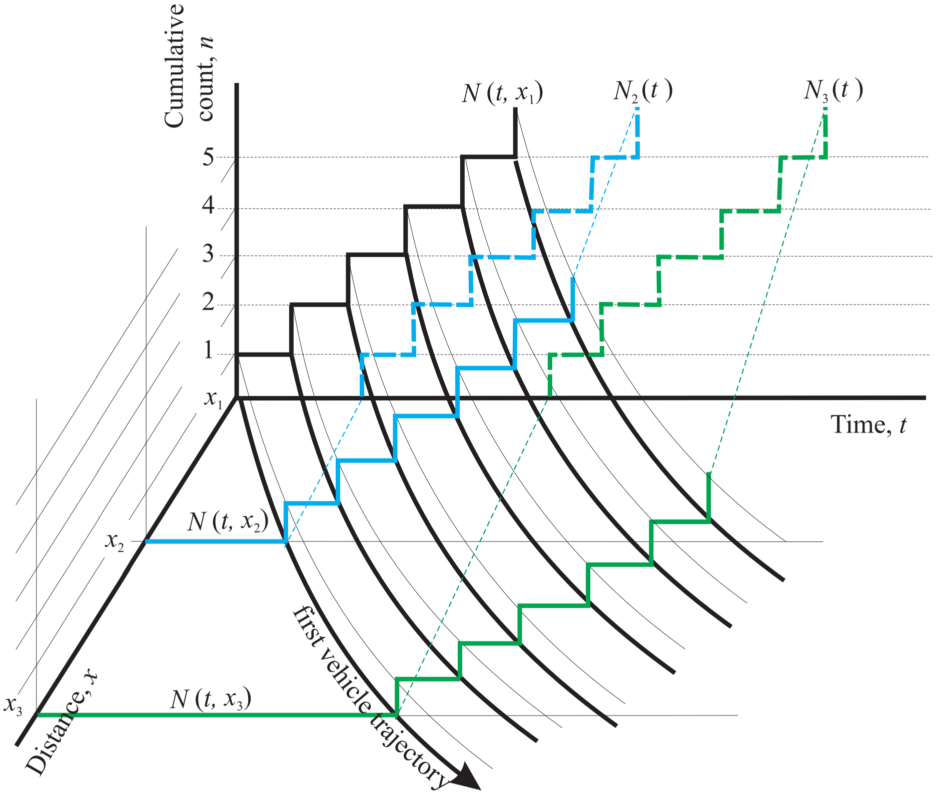
1. Determine the capacity of this freeway segment assuming that trucks arrive at the bottom of the uphill at fixed headways.
2. Exemplify your previous answer for: v1=60 km/hr, v2=80 km/hr, r1=r2=0.05 and L=1 km.



3. Cumulative count curves

3.1 Definitions: Delay, queuing, travel time

Recall the traffic flow surface.



Lecture Slides

Definition: Predictive delay

Definition: Reactive delay

R Unless Otherwise indicated, we will use predictive delay.

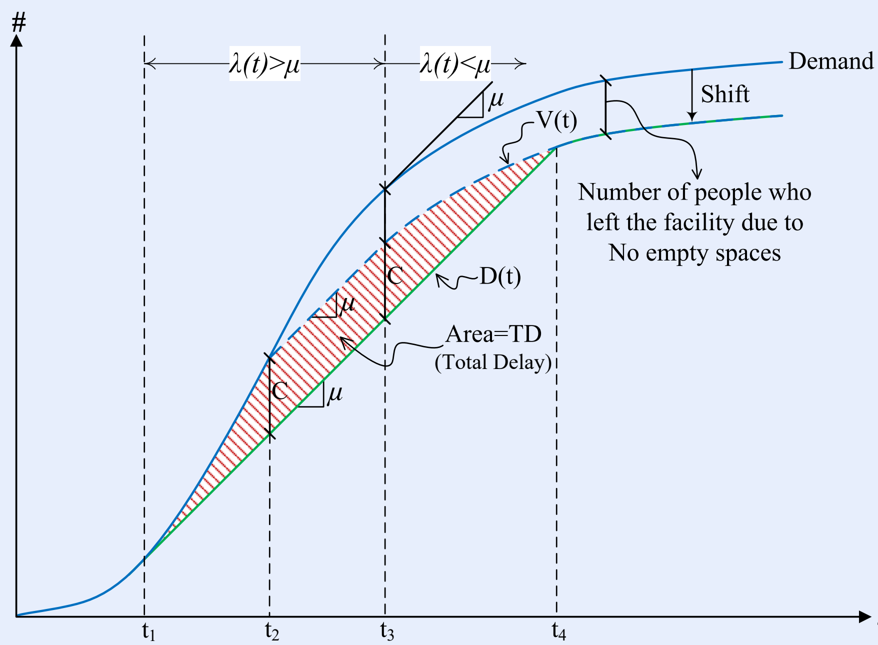
3.2 The Bottleneck model

Example 3.1. — Incident

Example 3.2. — Optimal freeway capacity

Example 3.3. — Diverge bottlenecks assuming FIFO

Example 3.4. — Limited storage capacity Customers bound for a coffee shop do not enter and leave the system when there are C customers already inside the store.



3.3 Dynamic Traffic Assignment

Definition: Traffic assignment is a procedure to allocate the demand between a given origin-destination to all possible alternative routes between them. Mainly two types: user equilibrium assignment and system optimum assignment.

Definition: User equilibrium A network is in user equilibrium (UE) when, either:

1. the routes being used between any origin and destination have the same cost; the unused routes have a higher cost, or

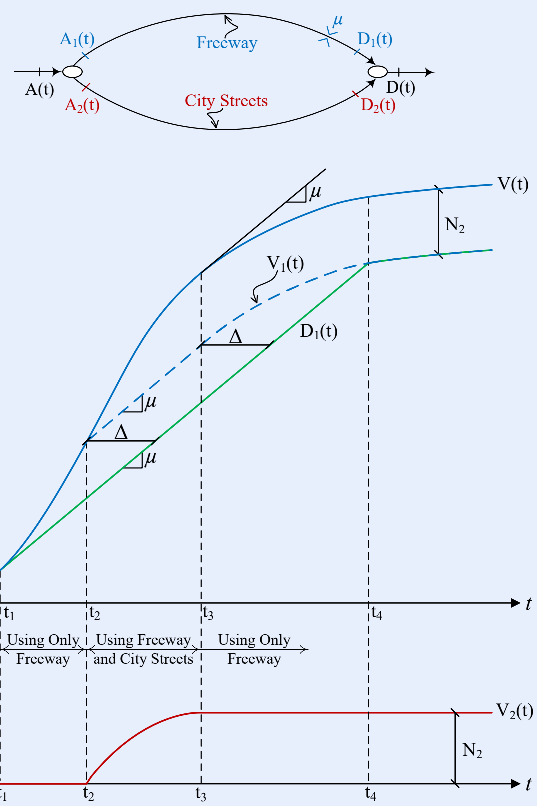
2. every driver chooses the route with the minimum travel cost,
3. rerouting a single driver cannot decrease travel cost.

Definition: System Optimum A network is in system optimum (SO) when the total system cost is the minimum among all possible assignments.

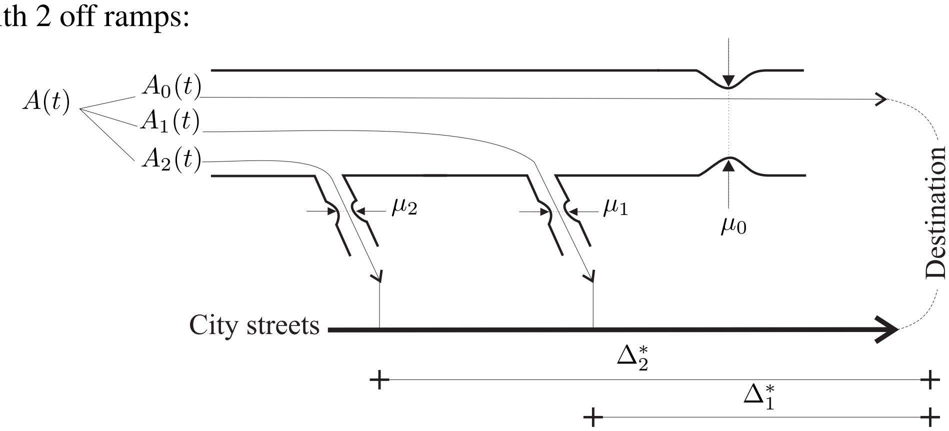
3.3.1 User Equilibrium

This section is based on this paper.

Example 3.5. — 2-Route UE-DTA There are two alternatives, the freeway with capacity μ and the city streets with unlimited capacity, but with an *extra* free-flow travel time Δ.



Now we examine the case where all alternatives have limited capacity. For example, consider a freeway with 2 off ramps:



Let $\Delta_r(t)$ be the trip time in route r at time t:

$$\Delta_r(t) = \Delta_r^* + w_r(t), \tag{3.1}$$

where Δ_r^* = free-flow travel time on off-ramp r and $w_r(t)$ = delay:

$$w_r(t) = \frac{A_r(t) - A_r(t_r)}{\mu_r} - (t - t_r), \qquad r = 0..R(t), \tag{3.2}$$

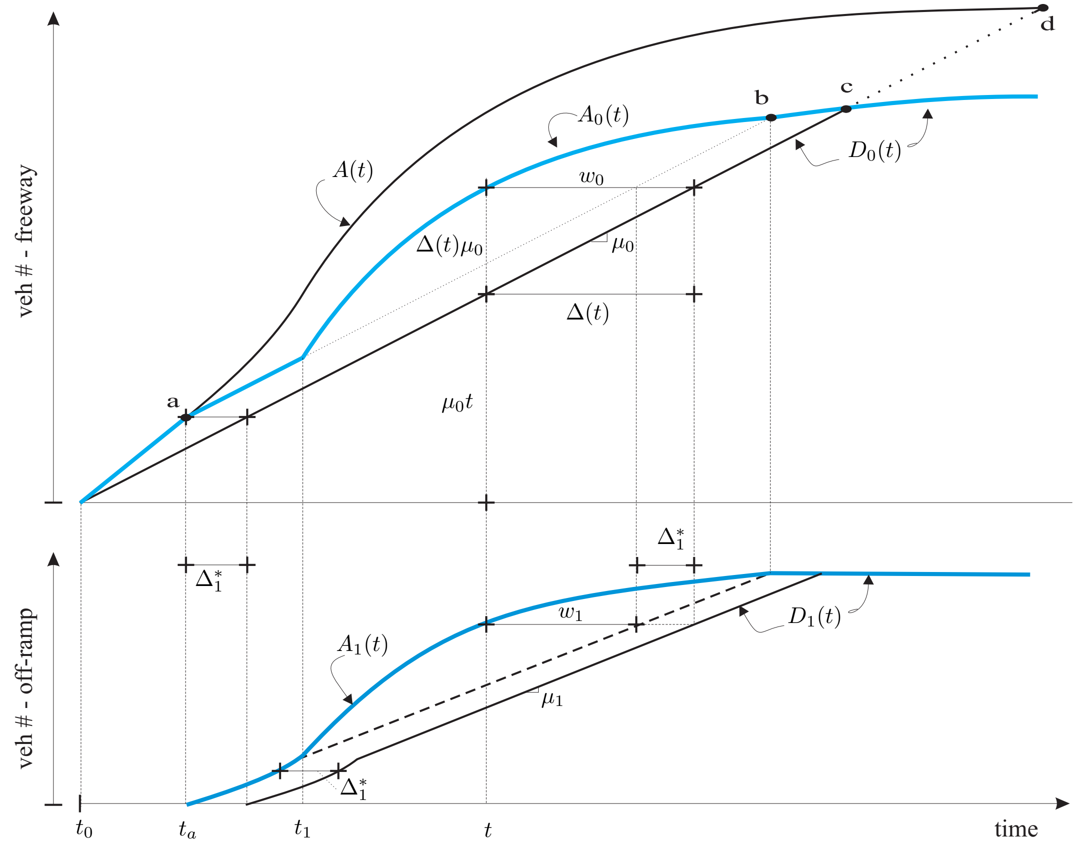
where t_r = the time when route r becomes congested, and $R(t)$ = most upstream off-ramp being congested by time t.

Under DUE equilibrium the travel time in equilibrium, $\Delta(t)$, satisfies

$$\text{UE at time } t \Rightarrow \begin{cases} \Delta(t) = \Delta_0(t) = \Delta_1(t) = \ldots = \Delta_{R(t)}(t) \leq \Delta_{R(t)+1}(t) \\ \Delta(t) < \Delta_{R(t)+2}(t), \Delta_{R(t)+3}(t) \ldots \end{cases} \tag{3.3}$$

Exact graphical solution

There are two assignment patterns, called type-1 and type-2 diversion patterns. In the case of a single offramp, cumulative arrivals can be constructed as shown below:



Type-1 diversion:

$$\dot{A}_r(t) = \begin{cases} \mu_r & \text{if } r \le R(t), \\ \dot{A}(t) - \sum_{k=0}^{R(t)} \mu_k & \text{if } r = R(t)+1, \\ 0 & \text{o/w.} \end{cases} \qquad \text{(type-1 diversion)} \tag{3.4}$$

For type-2 diversion patterns we use DUE in differential form; i.e.,

$$\dot{\Delta}_0(t) = \dot{\Delta}_1(t) = \ldots = \dot{\Delta}_{R(t)}(t). \tag{3.5}$$

The system of equations (3.2) and (3.5) leads to the following solution:

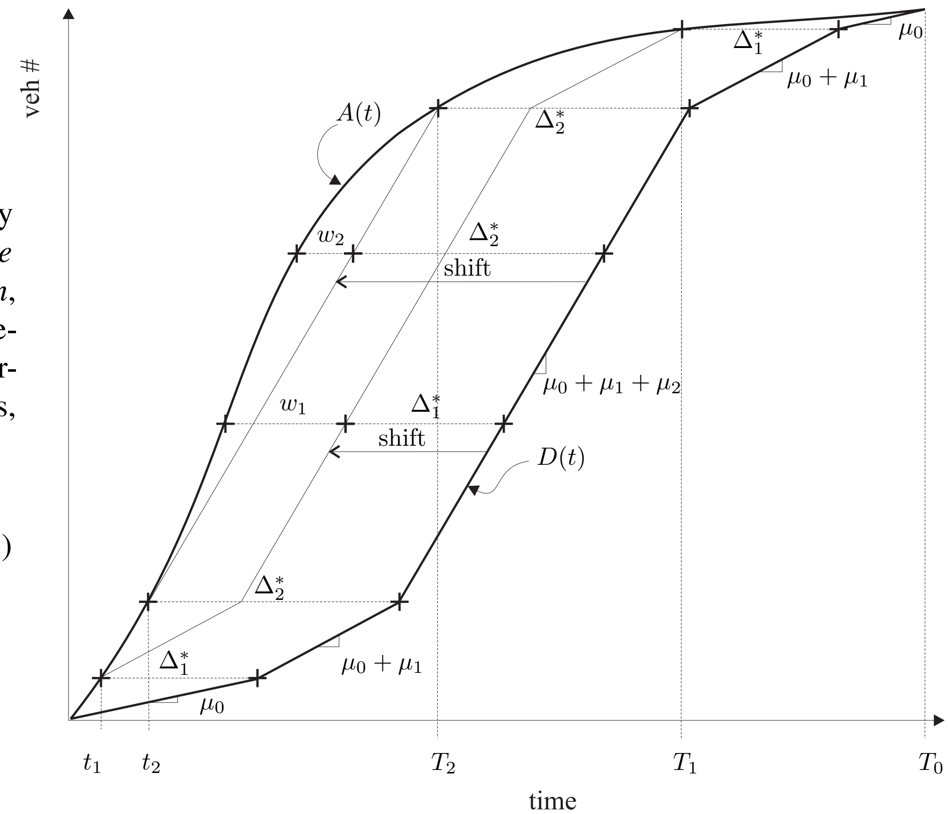
$$\dot{A}_r(t) = \begin{cases} \frac{\mu_r}{\sum_{k=0}^{R(t)} \mu_k} \dot{A}(t) & \text{if } r \le R(t), \\ 0 & \text{o/w.} \end{cases} \qquad \text{(type-2 diversion)} \tag{3.6}$$

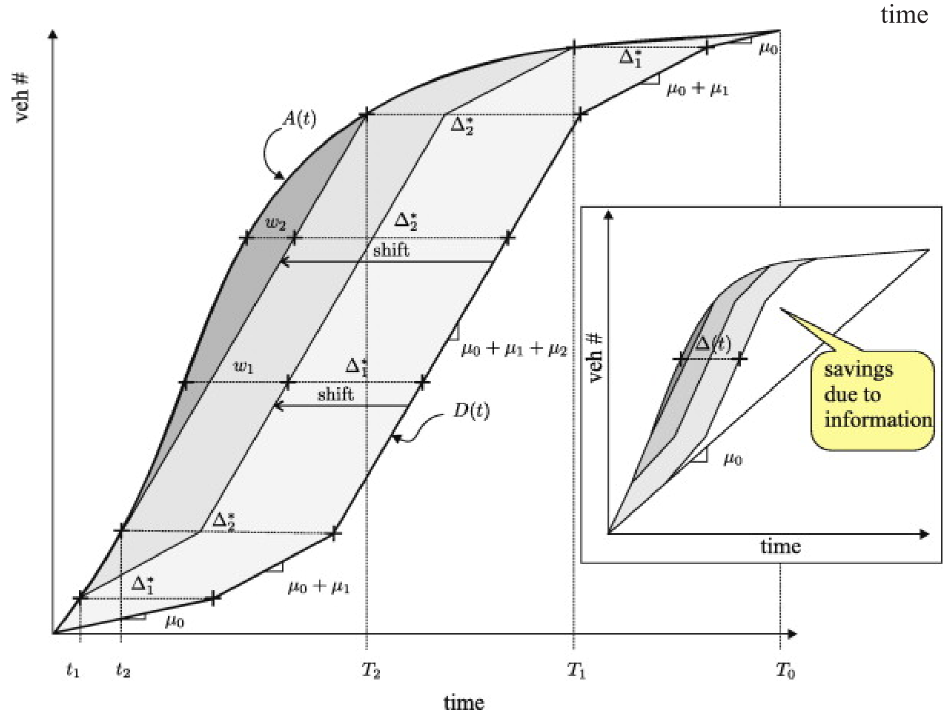
⇒ the flow using each route is a constant fraction of the total arrival flow, and this fraction is proportional to the capacity of the route.

Simplified graphical solution method

When arrivals increases rapidly at the beginning of the rush, *we can neglect Type-1 diversion*, and queuing delays can be presented graphically using total arrivals $A(t)$ and total departures, $D(t)$, defined as

$$D(t) = \sum_{k=0}^{R(t)} \mu_k. \tag{3.7}$$





3.3.2 System Optimum

This chapter is based on this paper. Lecture Slides

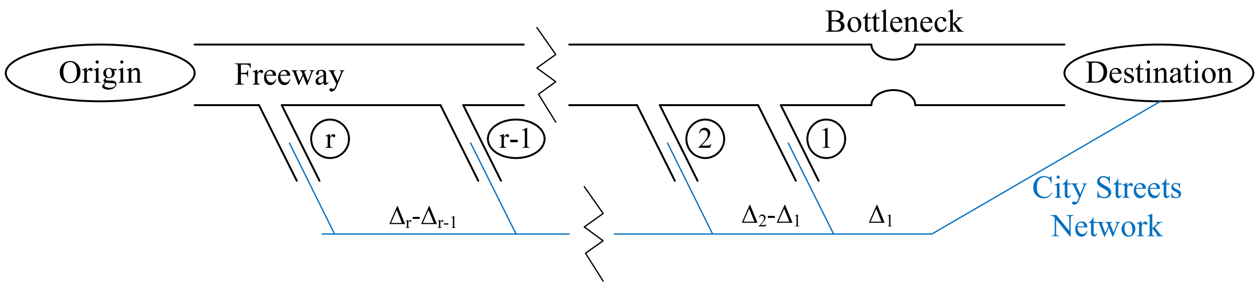


Figure 3.1: overview of the network

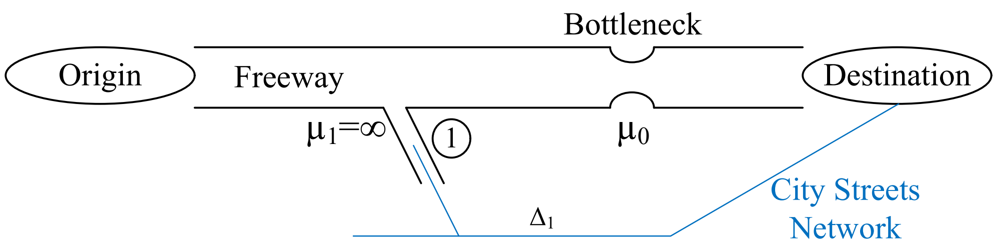


Figure 3.2: one off-framp with infinite capacity

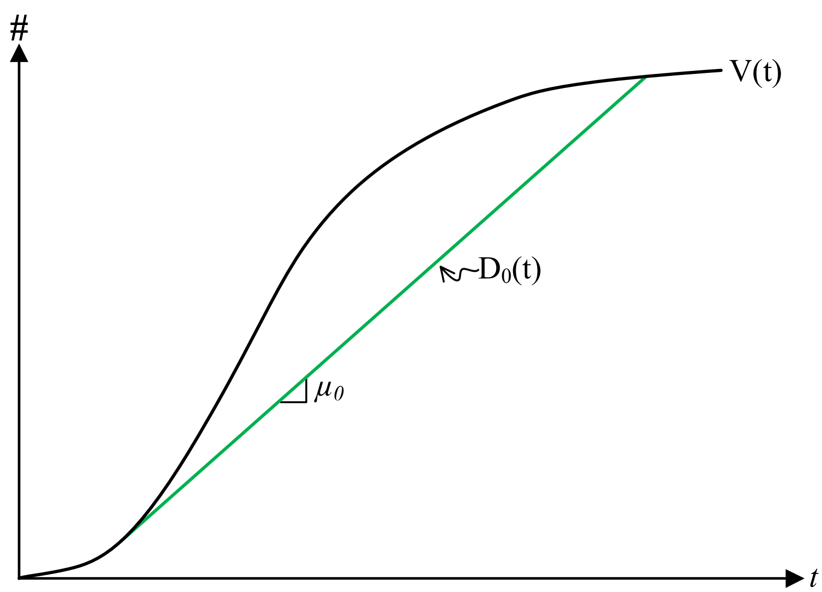


Figure 3.3: System Optimum 3

- given N, what is the best $V_0(t)$?
- what is the optimal N?

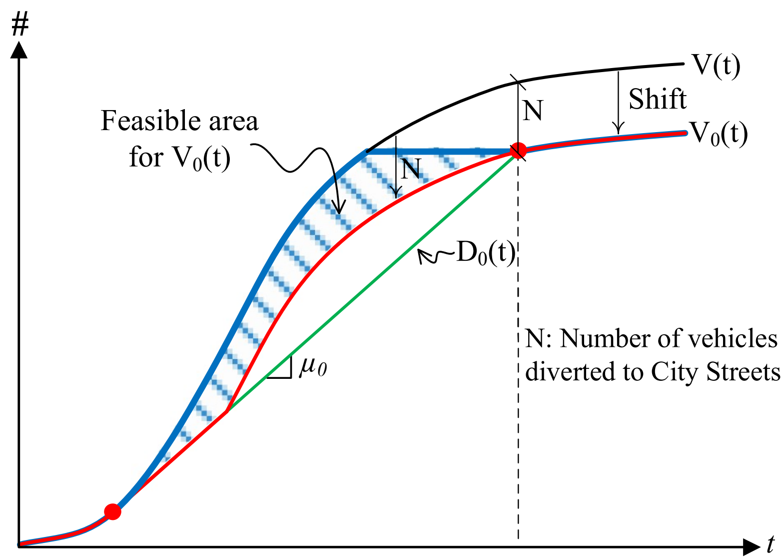


Figure 3.4: feasibility area for freeway arrivals

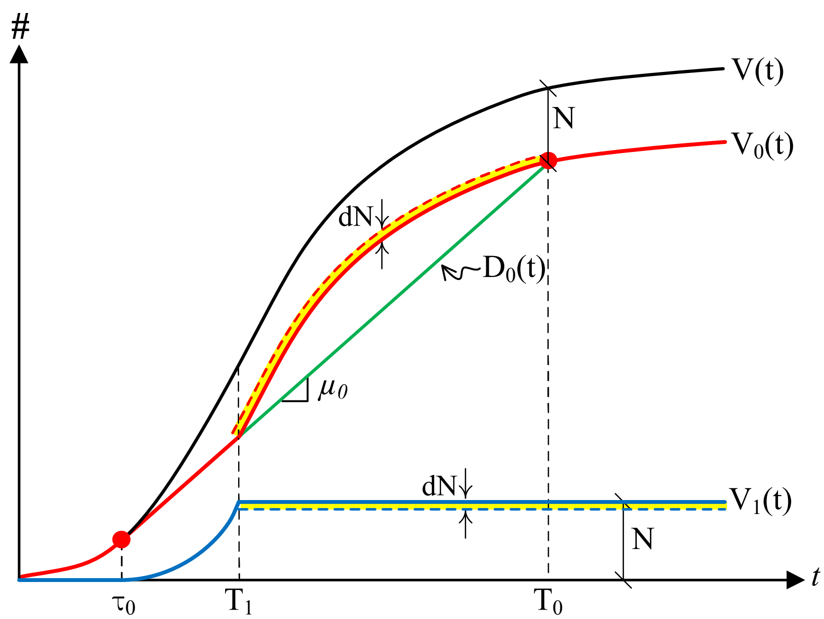


Figure 3.5: System Optimum 6

$$\boxed{T_0 - T_1 = \Delta} \quad \text{(Optimal solution)} \tag{3.8}$$

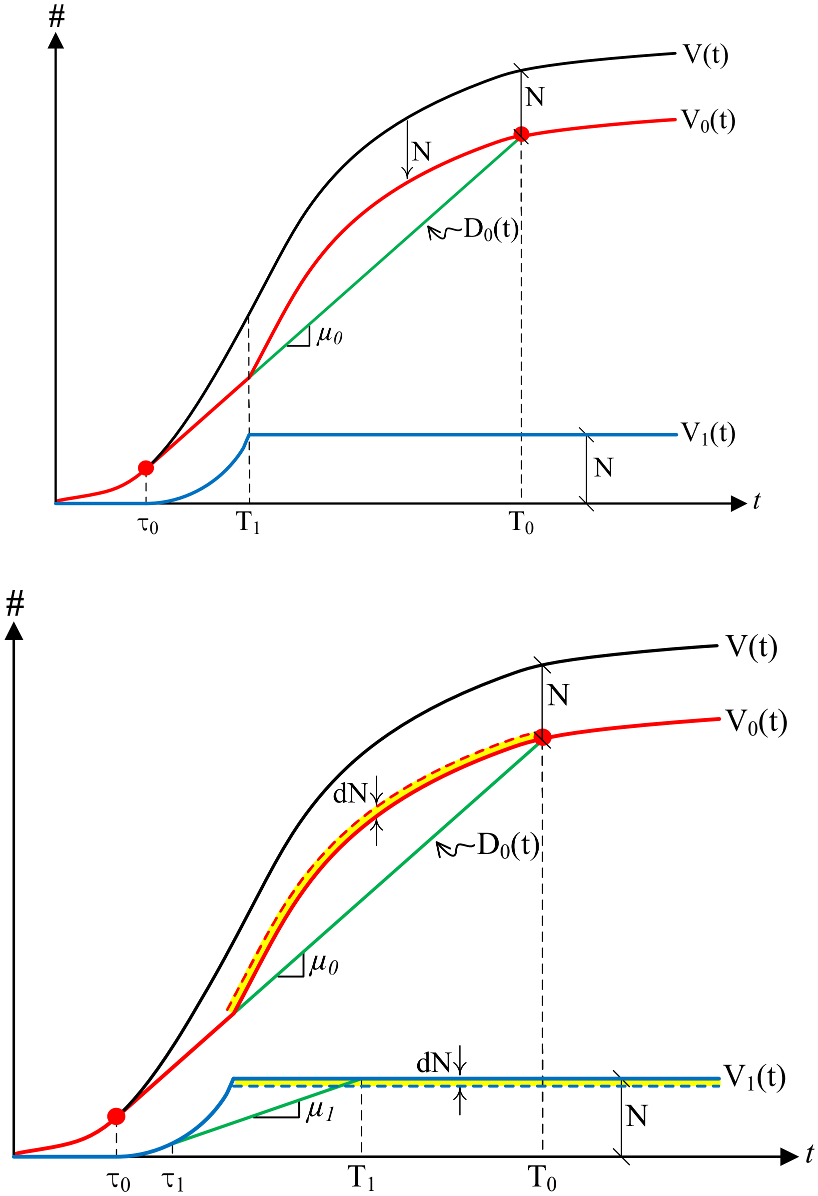


Figure 3.6: System Optimum 7

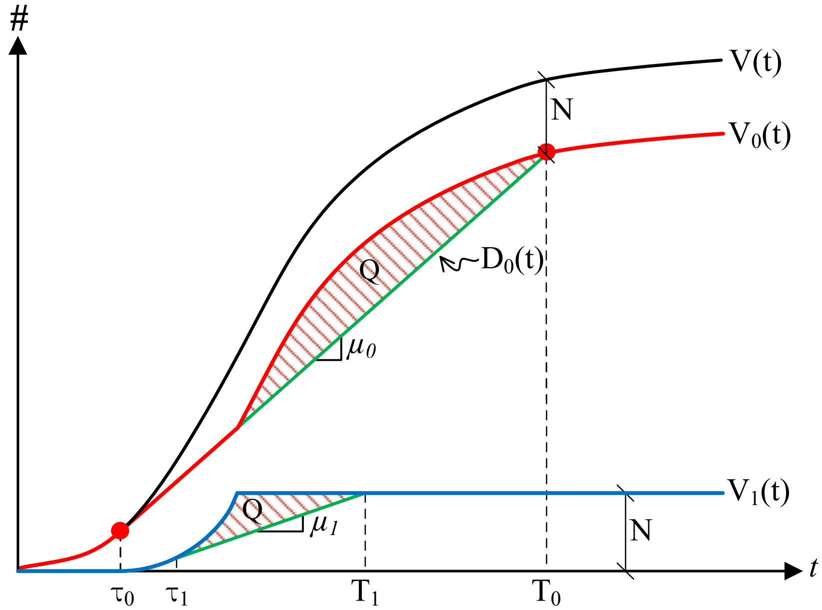


Figure 3.7: System Optimum 8

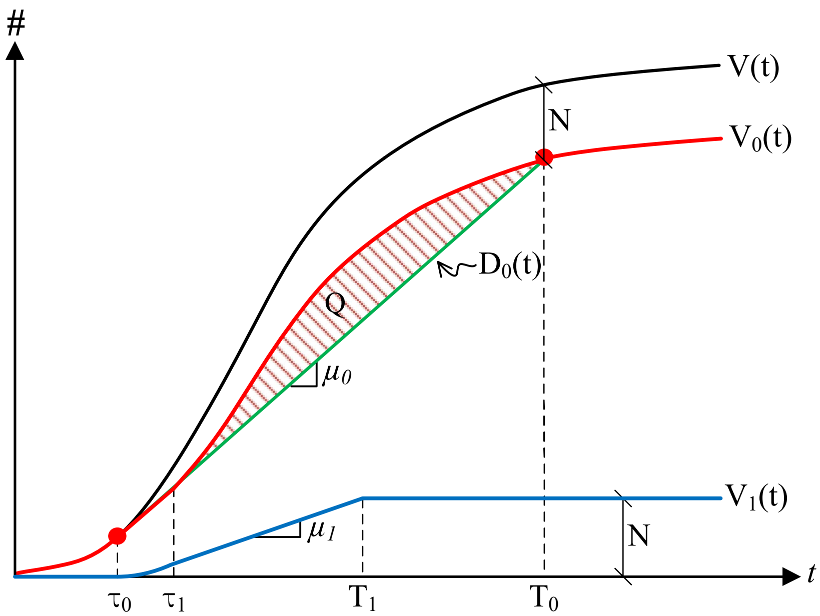


Figure 3.8: System Optimum 9

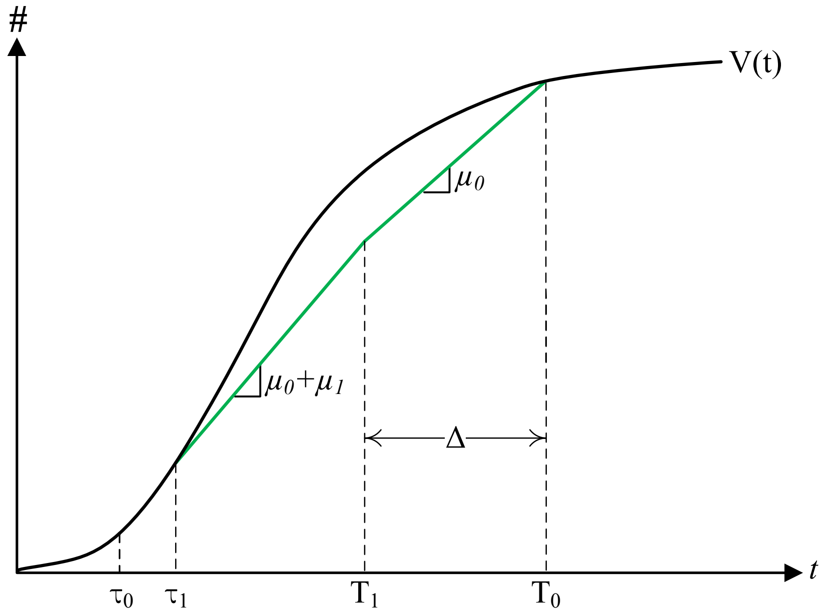


Figure 3.9: System Optimum 10

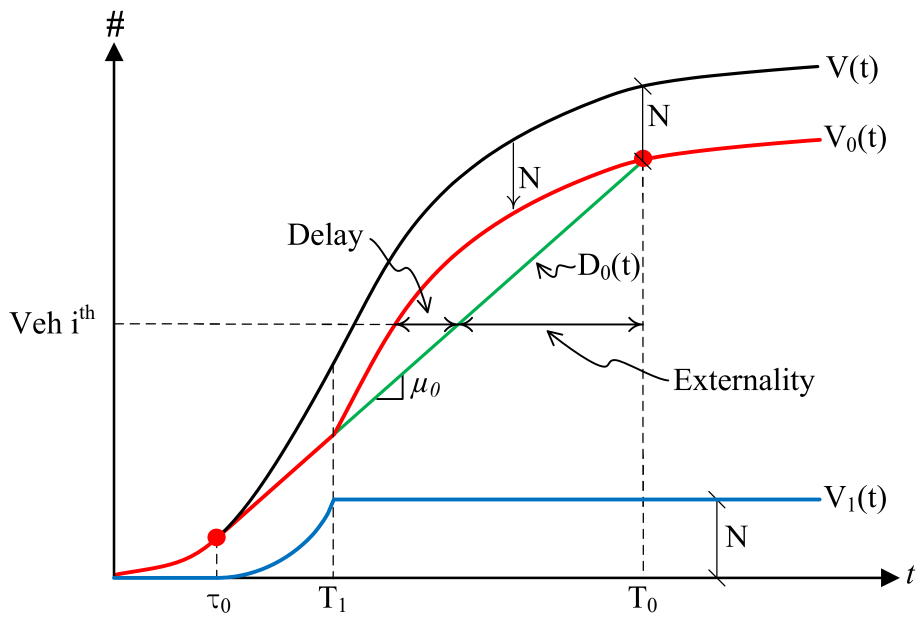


Figure 3.10: System Optimum 11

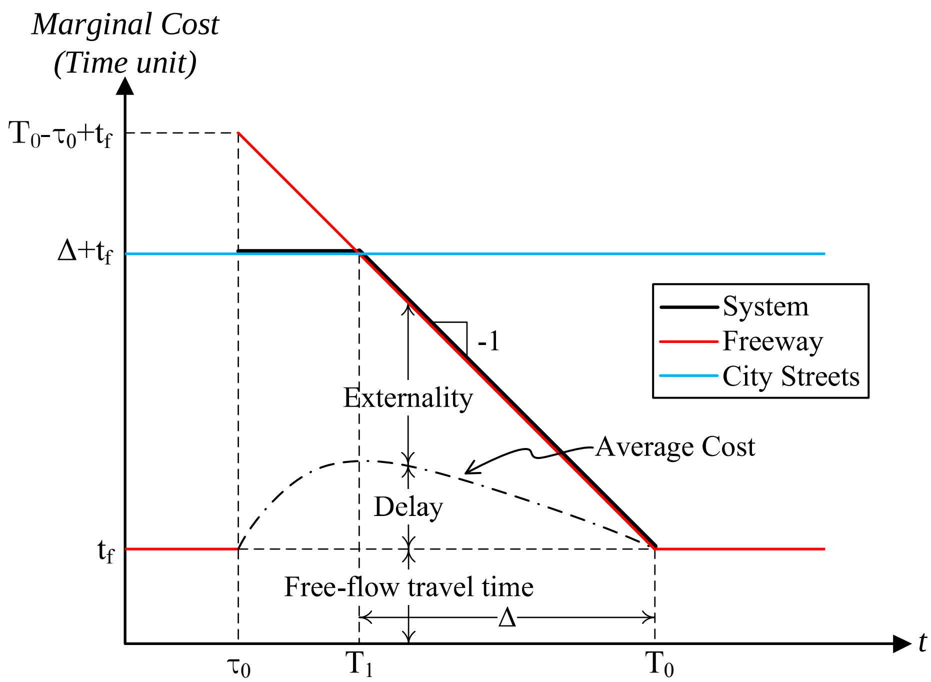


Figure 3.11: System Optimum 12

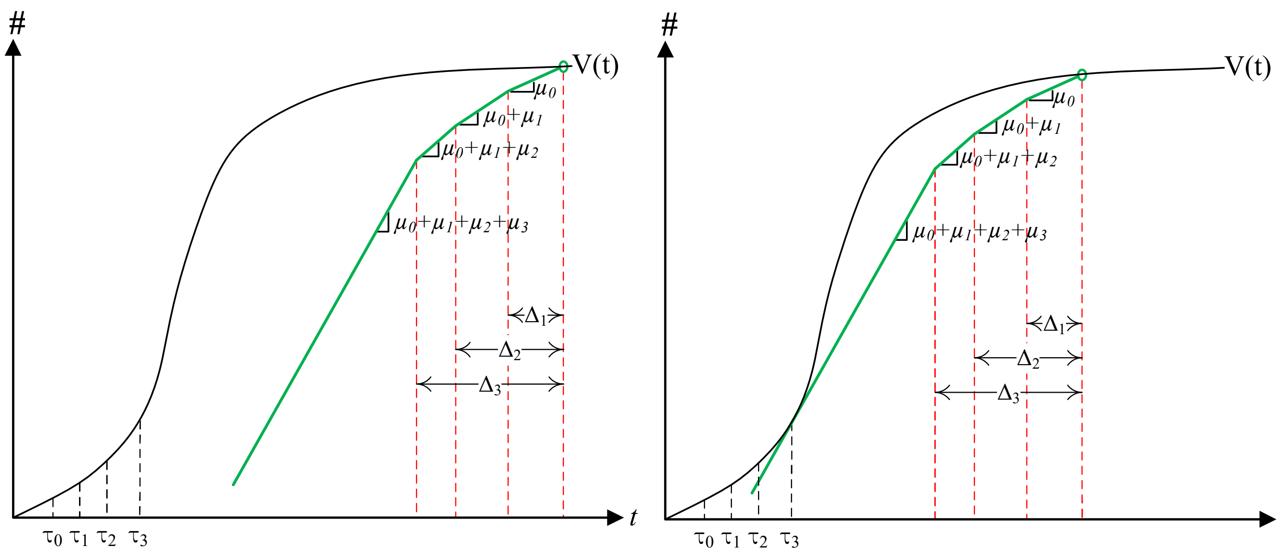


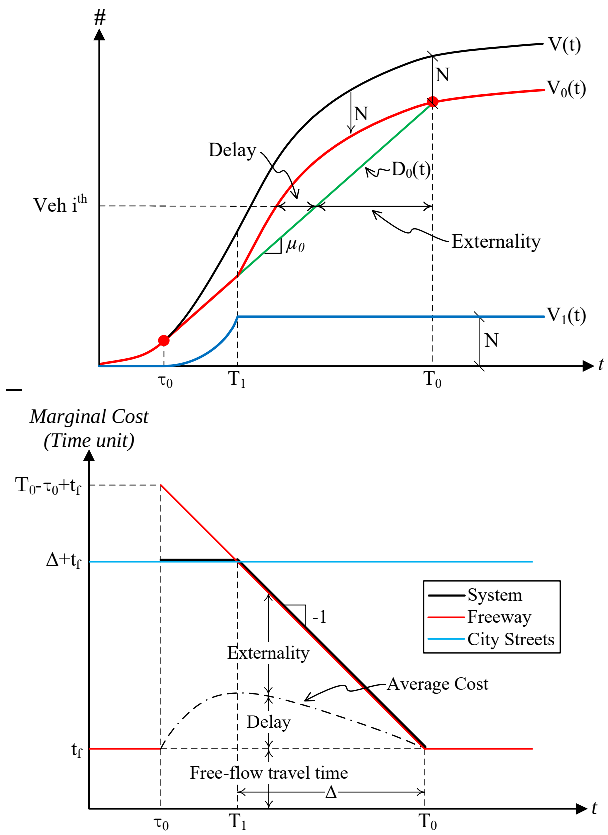
Figure 3.12: System Optimum 13

Example 3.6. — 2-Route SO-DTA There are two alternatives, the freeway with capacity μ and the city streets with unlimited capacity, but with an *extra* free-flow travel time Δ. ■

3.4 Congestion pricing

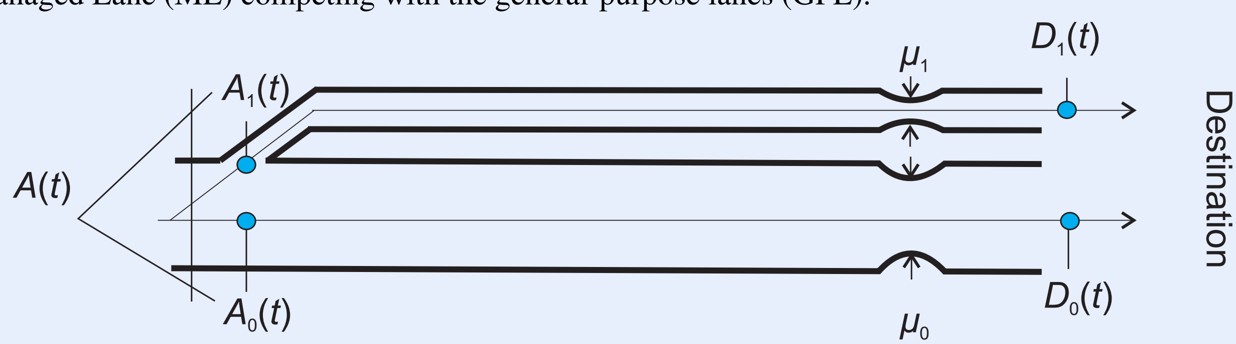
> **Main idea:** Find tolls to each alternative route such that under UE the resulting assignment is the SO.

We saw that under SO vehicles should take the minimum marginal cost route, which means that the SO is equivalent to the UE *if drivers perceived the marginal cost they produce*. This is the so-called first-best pricing: charge the user the equivalent of the marginal cost on each alternative route. Since this pricing mechanism is clearly unpractical, the **second-best** congestion pricing charges the difference in marginal cost to typically only a few alternatives, leaving the rest free.



Example 3.7. — Managed Lanes This example is taken from this paper, and could be easily extended to other cases such as toll roads or zone-based pricing.

Consider the equilibrium between two alternatives with finite capacity, one of which is priced: a Managed Lane (ML) competing with the general-purpose lanes (GPL).



Let $A(t)$ be the cumulative number of vehicles at time t that have entered a freeway segment containing a ML entrance, and let the corresponding flow be $\lambda(t) = \dot{A}(t)$. All vehicles are bound for a single destination past a GPL bottleneck of capacity μ_0, which may be bypassed by paying a toll $\pi(t)$ upon entering the ML at time t, which has a bottleneck of capacity μ_1.

The cumulative count curve of vehicles using route r (r=0 for the GPL and $r = 1$ for the ML) is denoted $A_r(t)$ and the flow, $\lambda_r(t) = \dot{A}_r(t)$. Clearly,

$$\lambda(t) = \lambda_0(t) + \lambda_1(t), \tag{3.9}$$

and is assumed unimodal. Let $\tau_r(t)$ be the trip time in route r experienced by a user arriving at time t:

$$\tau_r(t) = \tau_r + w_r(t), \tag{3.10}$$

where τ_r is the free-flow travel time, and $w_r(t)$ is the queuing delay, which can be expressed as:

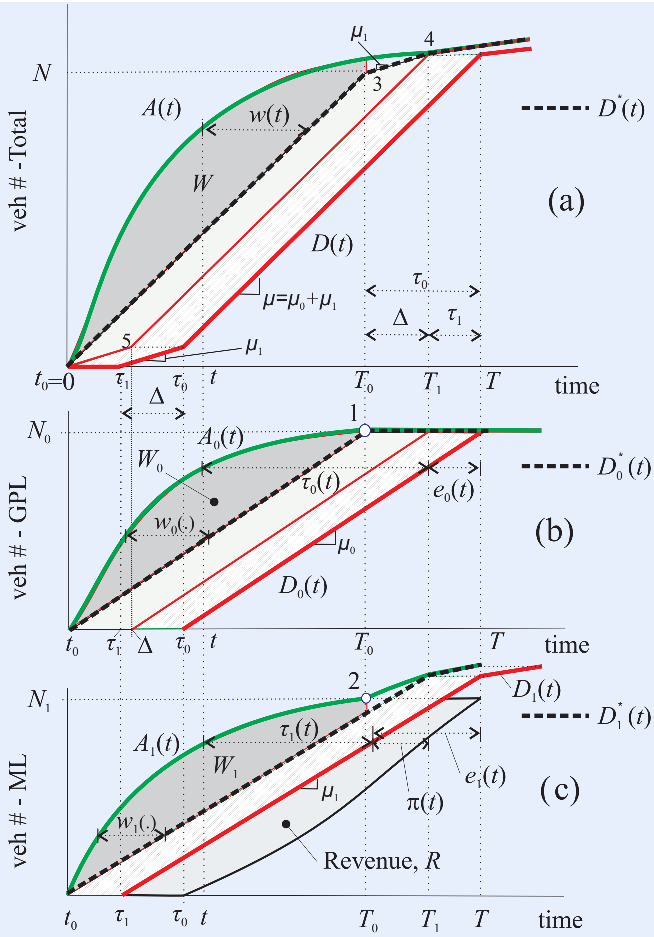
$$w_r(t) = \frac{A_r(t) - A_r(t_r)}{\mu_r} - (t - t_r), \qquad t_r < t < T_r, \tag{3.11}$$

where t_r and T_r represent the times when route r begins and ends being congested, respectively. Let:

$$\Delta = \tau_0 - \tau_1 \tag{3.12}$$

be the extra free-flow travel time for using the free alternative. Although in many cases one would expect $\tau_0 \approx \tau_1$, this will not be assumed here for maximum generality; we assume that $\Delta > 0$ which means that $t_1 < t_0$ in the SO solution, i.e. the ML is used at capacity before the GPL, as shown next.

In the figure below, total virtual departures $D^*(t)$ are defined as the arrival curve shifted to the right by the free-flow travel time τ_0. The area between these curves is the total system delay, i.e. the total time spent queuing in the system. The method to obtain the curve $D^*(t)$ was introduced in section 3.3.2, and is best visualized by imagining a ring connected to the rightmost end of $D^*(t)$ that is slid along $A(t)$ from right to left until $D^*(t)$ “touches" $A(t)$ again (at point “1" in the figure). This point corresponds to the time when both alternatives start being used at capacity (t_0 in our case since $\Delta > 0$, and $\lambda(t_0) = \mu_0 + \mu_1$), and from here one can identify the arrival time of the last vehicles to experience delay in each alternative, $T_r, r = 0, 1$, and the time when the shorter alternative starts being used at capacity, (t_1 in our case, and $\lambda(t_1) = \mu_1$). This figure also shows how to obtain the total system departure curve $D(t)$, which gives the count of vehicles reaching the destination at time t. Notice that total arrivals and departures in the system are not first-in-first-out.



The marginal cost $\tau_r(t)+e_r(t)$ in each alternative gives the extra cost incurred by the system if an additional unit of flow uses such alternative. In $t_0 \le t \le T_0$ the marginal cost is given by the time remaining until the end of congestion in the system, and it is identical on both alternatives, as expected. Outside this time interval only the alternative with the least marginal cost (ML in this case) is used.

The figure below shows the marginal cost in equilibrium along with travel times, delays, and externalities $e_r(t)$ on each alternative, as a function of time. The figure also shows the SO flow pattern in each relevant time interval, with the exception of $t_0 \le t \le T_0$, where SO flows are not unique, and nor are $\tau_r(t)$ and $e_r(t)$. It follows that in $t_0 \le t \le T_0$ the toll $\pi(t)$ is also not unique and can be chosen freely but within the following constraints:
(i) boundary conditions constraints:

$$\pi(t_0)=\Delta, \qquad \pi(T_0)=\Delta-w_1(T_0), \qquad \text{and} \tag{3.13a}$$

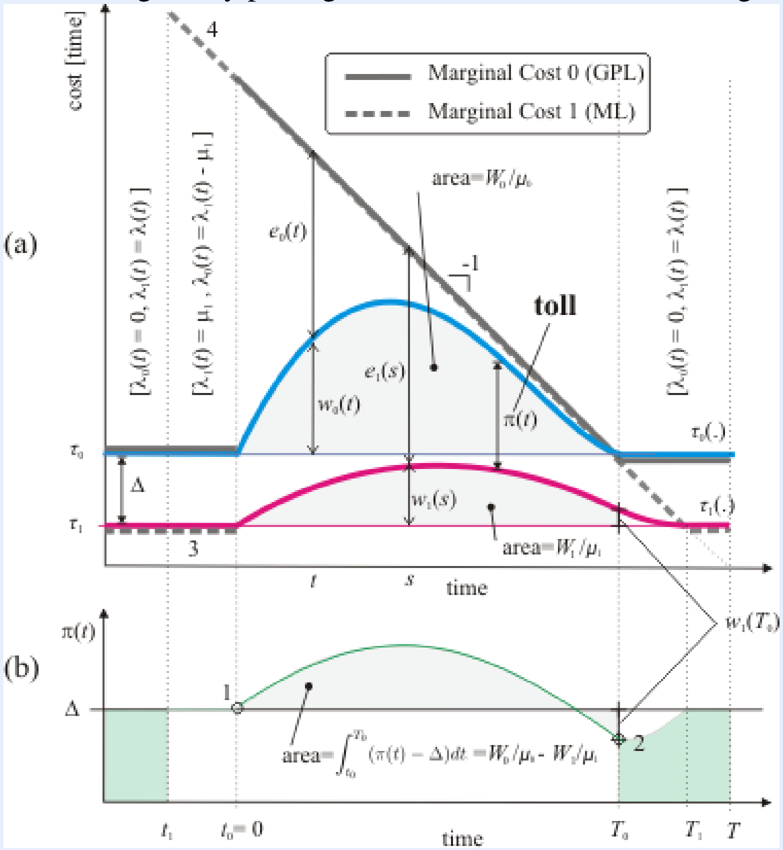
(ii) active bottleneck constraints:

$$\dot{\pi}(t) \ge (\mu-\lambda(t))/\mu_1, \qquad \text{if GPL at capacity with no queue} \tag{3.13b}$$

$$\dot{\pi}(t) \le (\lambda(t)-\mu)/\mu_0, \qquad \text{if ML at capacity with no queue} \tag{3.13c}$$

$$-\lambda(t)/\mu_1 \le \dot{\pi}(t) \le \lambda(t)/\mu_0, \quad \text{if GPL and ML have queues} \tag{3.13d}$$

The boundary condition constraints (3.13a)–depicted as points “1" and “2" in the Fig. above–are a consequence of the SO conditions in the time intervals $t \le t_0$ and $t \ge T_0$, which force pricing to be either fixed or arbitrary. Before t_1 there is no congestion and therefore as long as $\pi(t) \le \Delta$ all drivers will choose the ML, as required by the SO condition. After T_0 only the ML should be used, which can be achieved, again, by pricing within the shaded area in the figure.



The optimization of this system can be very simple, to the point where the appropriate pricing strategy to accomplish a given objective is reduced to choosing a single parameter value.

Definition: We say that tolls are linear in the arrivals if there is a constant, a, called the pricing coefficient, such that:

$$\pi(t) = (A(t) - \mu t)a/\mu, \qquad t_0 \leq t \leq T_0, \tag{3.14}$$

which means that the toll is proportional to the system queue at $t, A(t) - \mu t$, or delay $w(t) = (A(t) - \mu t)/\mu$. Notice that a is dimensionless. This strategy is "real-time" because from (3.14) it is clear that to determine the toll at time t all that is needed is the demand and capacity ratio at the same time, which can be measured in real-time. ■

3.5 Problems

Problem 3.1 — Incident Vehicles begin to arrive at a metered freeway on-ramp at 6 AM with an arrival rate of 1500 vph. The arrival rate decreases to 500 vph at 10AM and remains at this rate throughout a day. The maximum service rate at the ramp meter is 1000 vph.

a) Draw a queueing diagram that shows the arrival and the departure process. Label the arrival and the departure curves and the time that the queue dissipates.
b) Determine the time that the queue dissipates.

   On a particular day, an incident occurs at 7AM at the meter and is cleared an hour later. The accident completely blocks the on-ramp, allowing no vehicles to the freeway (i.e. no departure from the meter).

c) Draw the queueing diagram under this scenario and show the additional total delay incurred by the incident.
d) Determine the time that the queue dissipates on the day of the accident.
e) What is the additional duration of queued period and how does this compare to the duration of the incident?

Problem 3.2 — Airport queues* At an origin airport, passengers arrive at the ticket counters to check in their luggage and go through the security before they arrive to the departure gate. The time-dependent arrival rates of passengers to the ticket counters are 180 pax/hr for time 0 to 30 mins and 10 pax/hr afterwards. The capacities at the ticket counters and security checkpoints are 70 pax/hr and 35 pax/hr, respectively.

a) Calculate the total passenger delay at the ticket counters, between the ticket counters and security checkpoints, and for the entire system.
b) Calculate the change in total delay if you increase the capacity only at the ticket counters to 180 pax/hr.
c) Calculate the change in total delay if you increase the capacity only at the security checkpoints to 70 pax/hr and to 180 pax/hr.
d) In light of parts (b) and (c), what should you do to decrease the total delay by 50% and by 70%?

Problem 3.3 — Average delay across days Suppose that the customer cumulative (virtual) arrival curve at a bottleneck is of the form:

$$V(t) = Mid(0, (N_i/T)t, N_i) \quad \text{for} \quad N_i, T > 0$$

where the function Mid gives the middle value of three numbers (e.g. Mid(-1,3,2)=2), N_i is the total number of customers to be served during day i. If the bottleneck serves customers at a uniform rate μ every day, and N_i varies across days so that 50% of the days $(N_i/T) = 2\mu$ and 50% of the days $(N_i/T) = 3\mu$, determine:

(a) The total delay accumulated in the system after N days (N large), the average delay per day, and the average delay per customer over all the customers.
(b) Vavg, Davg, the area between these curves, and the average delay per customers one would estimate if all days were like the average day. Compare with (a).
(c) Derive the average across days of the average delay per customer.

Problem 3.4 — Airport capacity An airport runway has a capacity of 60 operations (i.e., landings & take-offs) per hour. In a particular hour, there are 30 landings evenly spaced over the hour. During that same hour, 20 departing aircraft uniformly arrive at the runway within a 20-minute period. Assume no other departing aircraft arrive at the runway during the hour.
a) What is the maximum number of aircraft waiting during the hour?
b) Assume that landing aircraft have priority over departing aircraft. Also assume that each aircraft carries 200 passengers. What is the total delay to the passengers in the 20 departing aircraft?

Problem 3.5 — Toll plaza* A toll plaza is capable of serving cars at a constant rate of μ cars/hr. Cars arrive at a constant rate $\lambda_1 < \mu$ until some time t = 0 (e.g. 7:00 a.m.), but from t = 0 until some time τ, they arrive at a rate $\lambda_2 > \mu$ (where λ_1 and λ_2 are in units of cars/hr). After time τ, the arrival rate returns to the value λ_1 and remains there until time τ' (e.g. 07:00 a.m. the next day) when the pattern repeats itself. Vehicles not served immediately form a queue and are served first-in, first-out. Draw curves for the cumulative (virtual) arrivals and departures of vehicles starting at time t = 0. Evaluate (i.e., derive in terms of the variables given) the following:

a) the maximum queue length, the longest delay to any customer, the duration of the queue, and the total delay to all vehicles during the time 0 to τ'.

b) Imagine that the peak period in the above problem (i.e., t = 0 to t = τ) occurs from 7:00 a.m. to 9:00 a.m. If the cost of providing round the clock service at the toll plaza, *C ($/day)*, is proportional to the toll plaza capacity, μ *(cars/hr)*, $C = \beta\mu$, and if each customer hour of delay is valued at δ *($/hr)*, find an expression for the capacity level that will minimize the total cost per day.

Problem 3.6 — Simplified graphical solution, UO and SO Consider the cumulative count curve A(t) of vehicles entering a freeway segment with a bottleneck of capacity μ_0 = 4000 vph. Upstream of the bottleneck there is one off-ramp per km, each with capacity 1000 vph. The average speed on the arterials is 30 kph. Consider a demand such that the first 30 minutes the arrival flow is 10,000 vph and then decreases to 2000 vph.

a) plot to scale the simplified graphical solution method for the user equilibrium,
b) plot the solution for the system optimum,
c) compare the total delay of each assignment method and comment.

Problem 3.7 — A ramp meter * is being considered at an entrance to a freeway, such that the maximum flow that could enter the freeway is μ. Currently, rush hour traffic arrives at the on-ramp at a rate λ_1, from time $t = 0$ to time $t = t^*$, and the at a (lesser) rate λ_2. Assume that $\lambda_2 < \mu < \lambda_1$.

(a) Assuming that drivers will not change their trips, draw and label a queuing diagram showing (virtual) arrivals and departures. Label the maximum delay experienced by any vehicle (Wmax).

(b) If an alternate route is available to drivers, and if they will take this route if their "predictive" delay at the ramp meter is greater than Wmax/2, add this new scenario to your diagram. Now, show graphically the following:

1. The number of vehicles which will divert.
2. How much earlier the queue will dissipate (compared to part (a)).

(c) Repeat part (b) using "reactive" delay and compare your results.

(d) Under system optimum would you expect more or less diversion? Why?

Problem 3.8 — HOT lane – UO A long n-lane freeway corridor currently has a bottleneck which reduces its capacity by one lane. That is, if the capacity of one lane is Q, the bottleneck capacity is $\mu = (n-1)Q$. A HOT lane is planned in the median lane that will be barrier-separated from the general-purpose (GP) lanes and will replace the left-most GP lane. Thus, the new GP bottleneck capacity $\mu_1 = (n-2)Q$. It is expected that the HOT bottleneck capacity will be $\mu_0 = 0.8Q$ due to weaving. The free-flow speed in the GP and HOT lanes are identical, there is a fixed toll r for using the HOT lane and drivers value of time is $\alpha\$/hr$. Total demand arrives at a rate λ_1, from time $t = 0$ to time $t = t^*$, and the at a (lesser) rate λ_2. Assume that $\lambda_2 < \mu_0 + \mu_1 < \lambda_1$.

Assuming user equilibrium and predictive delay, determine:

a) the "before" total delay
b) the "after" total delay
c) under what conditions will there be system benefits for the user.
d) repeat (b) and (c) above if the toll is time-dependent and proportional to the queue in the HOT lane, $q_1(t)$; i.e., r(t) = b*$q_1(t)$.
e) repeat (b) and (c) above if the toll is time-dependent and proportional to the queue in the GP lanes, $q_0(t)$; i.e., r(t) = b*$q_0(t)$.

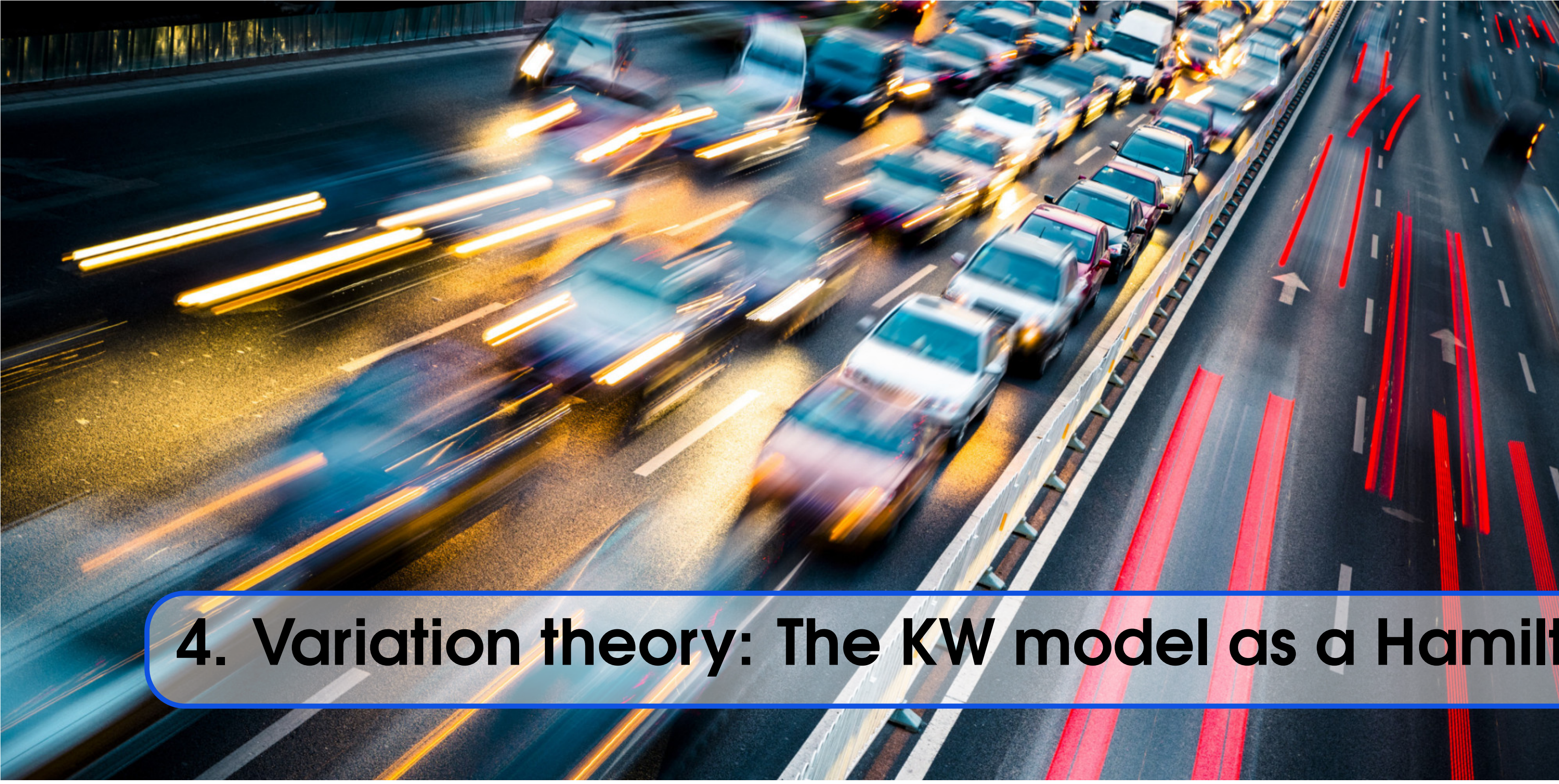
Problem 3.9 — HOT lane – SO Repeat the above problem assuming system optimum equilibrium.

Problem 3.10 — 2-route DTA Consider the capacitated 2-route dynamic traffic assignment problem seen in class. The capacity on the city street route is μ_1 and on the freeway μ_0. Total demand arrives at a rate λ_1, from time $t = 0$ to time $t = t^*$, and the at a (lesser) rate λ_2. Assume that $\lambda_2 < \mu < \lambda_1$. Prove that:

a) the number of drivers diverted to the city streets under SO is greater than under UO.
b) savings in total delay under SO are greater than under UO.

Problem 3.11 — diverge BN* The demand on a three-lane freeway is 6,500 vehicles per hour. It has been observed that the proportion of vehicles wanting to exit the freeway at a given off-ramp is 15%. The 200 m single-lane off-ramp ends at a traffic signal, which is green 50% of the time. An incident takes place on the off-ramp just downstream of this split , which reduces the capacity (at the incident location) to 500 vehicles per hour during 20 minutes.

a) Determine the total and average delay incurred by exiting and non-exiting drivers due to the incident.



4. Variation theory: The KW model as a Hamilt

The Hamilton-Jobi representation of the kinematic wave model allows us to use variational methods for its solution. Recall that

Definition: Cumulative count curves, $N(t,x)$, give the cumulative number of vehicles that have crossed location x by time t, and

$$q = N_t, \quad k = -N_x. \tag{4.1}$$

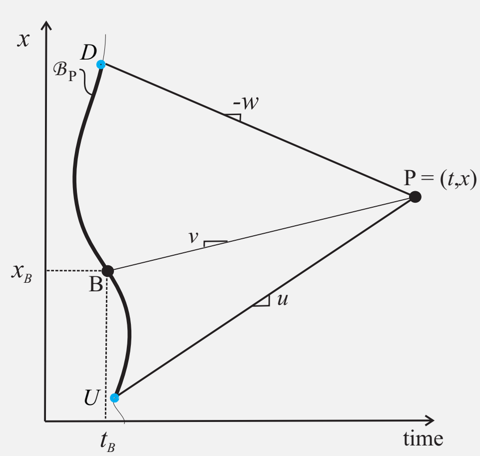
The Hamilton-Jacobi PDE expresses the existence of the fundamental diagram. For a homogeneous segment we have $q = F(k)$, or:

$$\begin{cases} N_t - F(-N_x) = 0, & \text{(fundamental diagram)} \quad (4.2a) \\ N(B) = G(B), \quad B \in \mathscr{B}, & \text{(boundary conditions)} \quad (4.2b) \end{cases}$$

which is a Hamilton-Jacobi PDE. The solution of (4.2) at a generic point $P \equiv (t,x)$ is the so-called *Hopf-Lax* formula:

$$N(P) = \min_{B \in \mathscr{B}_P} \left\{ G(B) + (t - t_B) R\left(\frac{x - x_B}{t - t_B}\right) \right\} \tag{4.3}$$

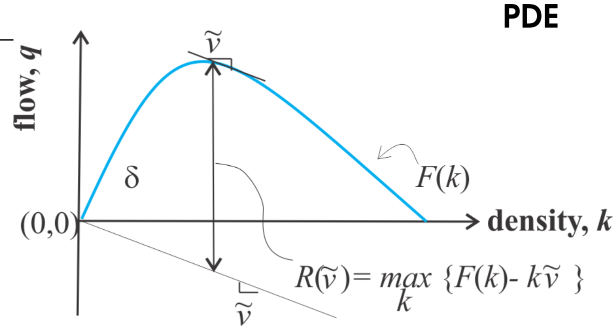
where $B \equiv (t_B, x_B)$ is a point in the boundary $\mathscr{B}_P$. The solution (4.3) is a **shortest path problem**: find point B on the boundary so that the cost from P to B is minimized.



Definition: Maximum passing rate The function $R(\cdot)$ gives the maximum passing rate along the observer and corresponds to :

$$R(v) = \sup_{k} \{F(k) - vk\}, \tag{4.4}$$

and is defined only for observer speeds $v \in [-w, u]$.



Example 4.1. Show that for the case of a triangular fundamental diagram we have:

$$R(v) = Q - K_c v \tag{4.5}$$

and,

$$R(u) = 0, \quad \text{and} \quad R(-w) = wK \tag{4.6}$$

■

4.1 Solution methods for triangular fundamental diagram

In the case of a triangular fundamental diagram the solution to (4.3) simplifies to:

$$N(P) = \min_{B \in \mathscr{B}_P} \{G(B) + (t - t_B)Q - (x - x_B)K_c\}. \quad (4.7)$$

It will be convenient to define the function to minimize:

$$f(B) \equiv G(B) + (t - t_B)Q - (x - x_B)K_c \quad (4.8)$$

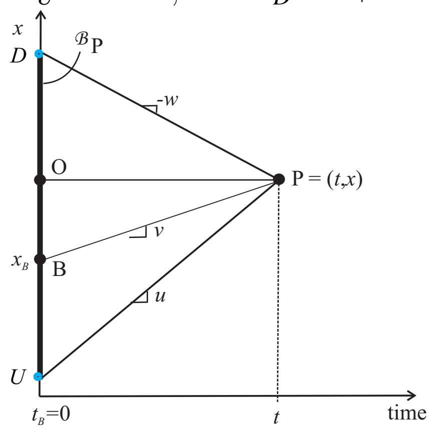
and our problem can be written as:

$$N(P) = \min_{B \in \mathscr{B}_P} f(B) \quad (4.9)$$

Now we will study how this solution is simplified for initial value problems (IVP) and boundary value problems (BVP).

4.1.1 Initial value problems

In the initial value problem (IVP) the boundary $\mathscr{B}$ is the line $\{t_B = 0\}$, so that the problem here is to minimize $f(x_B) \equiv G(x_B) + tQ - (x - x_B)K_c$ supplemented with $N(0,x) = G(x)$. The candidate set for B is $\mathscr{B}_P$ and reduces to B's x-coordinate, x_B, which is delimited by two points $U = (0, x_U)$ and $D = (0, x_D)$, where $x_U = x - ut$, and $x_D = x + wt$. Therefore, we have the following definition:



Definition: IVP

$$N(P) = \min_{x_U \le x_B \le x_D} f(x_B) \equiv G(x_B) + tQ - (x - x_B)K_c, \quad (4.10a)$$

$$N(0,x) = G(x) \quad (4.10b)$$

$$x_U = x - ut, \text{ and } x_D = x + wt, \quad (4.10c)$$

The first and second order conditions for a minimum in this case are

$$0 = f'(x_B) = G'(x_B) + K_c = -k(0, x_B) + K_c \quad (4.11a)$$

$$0 < f''(x_B) = G''(x_B) = -k_x(0, x_B) \quad (4.11b)$$

where we assume that G is at least twice-differentiable.

IVP Solution The set of candidates for the IVP is given by:

1. all x_B's in $x_U < x_B < x_D$ satisfying $k(0, x_B) = K_c$ and $k_x(0, x_B) < 0$.
2. extreme points x_U and x_D

The solution can be expressed as:

$$N_P = \min\{f(x_U), f(x_D), f(x_B^1), f(x_B^2), \ldots\}. \quad (4.12)$$

where $x_B^1, x_B^2, \dots$ are the candidates satisfying case 1 above.

Case 1 above happens when the IVP involves an acceleration from congestion to free-flow; case 2 is necessary because the domain of x_B is bounded. For instance, if the initial data is strictly congested, i.e. $k(0, x_B) > K_c, \forall x_B$, then $f'(x_B) < 0$ and the minimum will be attained at the largest possible value of x_B, which is x_D.

Example 4.2. — IVP Consider an homogeneous road segment with a triangular fundamental diagram where $w = -20$ km/hr, $K = 150$ veh/km and $u = 100$ km/hr. The initial density given by

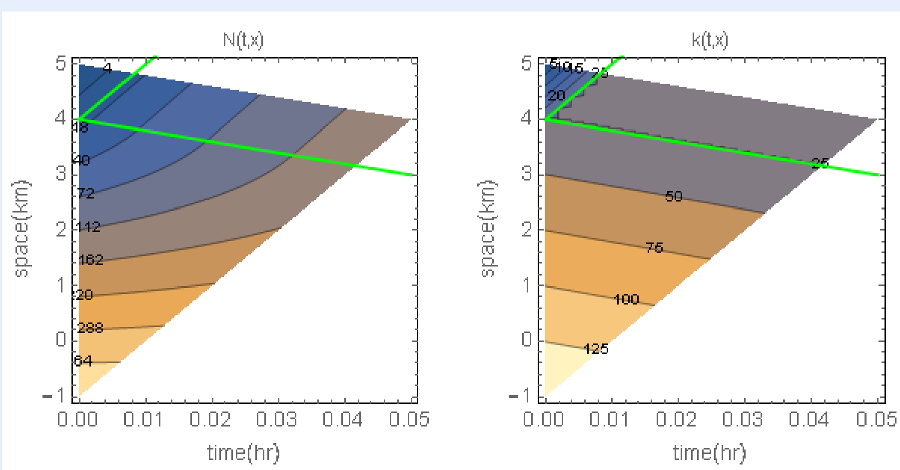
$$k_0(x) = K - K_c(1 + x), -1 \leq x \leq 5$$

solving for the first-order condition $k_0(x) = K_c$ gives $x_B^1 = 4$. $k_0(x)$ can be integrated to obtain

$$G(x) = 12.5x^2 - 125.x + 312.5$$

$$
\begin{aligned}
N_P &= \min\{f(x_U), f(x_D), f(x_B^1)\}, \\
f(x_U) &= G(x_U) = 312.5 + 25.(-5.(-100.t + x) + 0.5(-100.t + x)^2), \\
f(x_D) &= G(x_D) + K(x_D - x) = 25.\left(0.5(20.t + x)^2 - 5.(20.t + x)\right) + 3000.t + 312.5, \\
f(x_B^1) &= G(x_B^*) + (t - 0)Q - (x - 4)K_c = 2500.t - 25.(x - 4.) + 12.5.
\end{aligned}
$$

Notice that since this is an acceleration problem, the acceleration fan is delimited by the free-flow and congestion characteristics shown as green lines in the figure below, whose equations are $x = 4 + 100t$ and $x = 4 - 20t$, respectively. One can verify that these equations satisfy $f(x_D) = f(x_B^1)$ and $f(x_U) = f(x_B^1)$, respectively.



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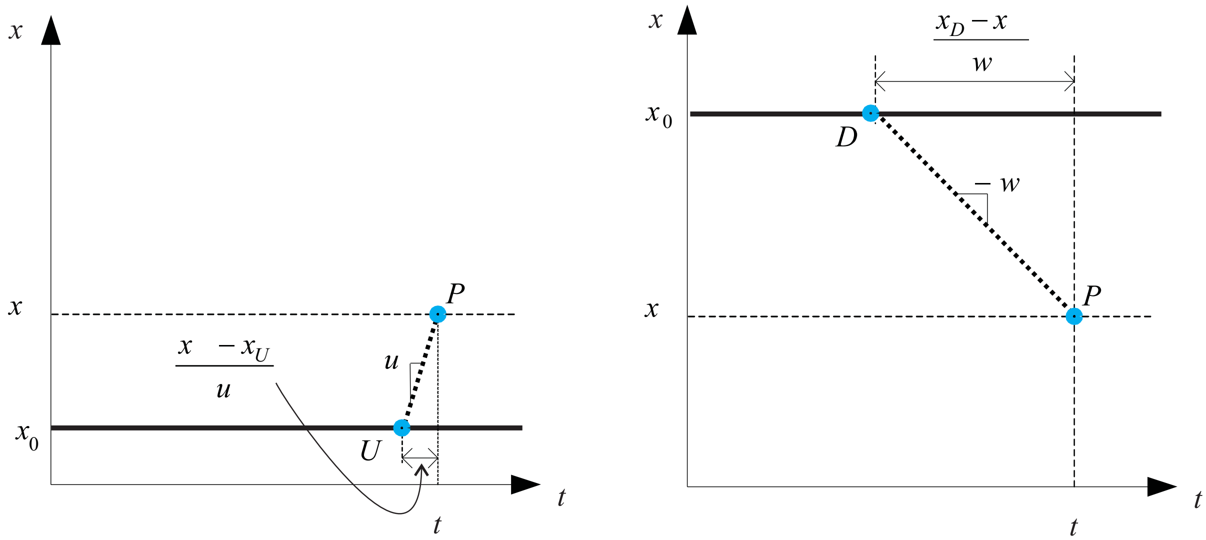
IVP Solution with linear data If the initial data $G(x)$ is linear then the solution can be expressed as:

$$N_P = \min\{f(x_U), f(x_D)\}. \tag{4.14}$$

4.1.2 Boundary value problems



In the boundary value problem (BVP) the boundary $\mathscr{B}$ is the line $\{x_B = x_0\}$; Similarly as in the previous section, we have now:

Definition: BVP If $x > x_0$ the variational problems is:

$$N(P) = \min_{0\le t_B \le t_U} f(t_B) \equiv G(t_B) + (t - t_B)Q - (x - x_0)K_c, \tag{4.15a}$$

$$N(t, x_0) = G(t) \tag{4.15b}$$

$$t_U = t - (x - x_0)/u \tag{4.15c}$$

The first and second derivatives are:

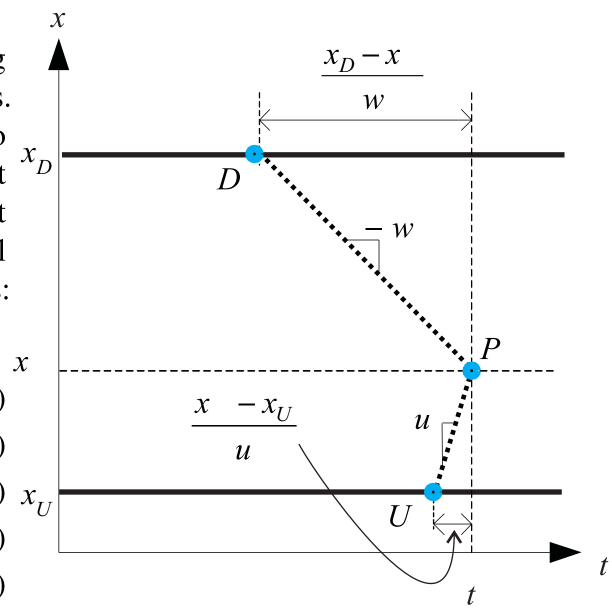
$$f'(t_B) = G'(t_B) - Q = q(t_B, 0) - Q \tag{4.16a}$$

$$f''(t_B) = G''(t_B) = q_t(t_B, 0) \tag{4.16b}$$

It can be seen that $f'(t_B) \le 0$ (Because flow cannot exceed capacity), and therefore the optimal candidate has to be t_U.The reader can verify that

Set of candidates for the BVP It is given by:
1. $t_U = t - (x - x_0)/u$ if $x > x_0$,
2. $t_D = t - (x_0 - x)/w$ if $x < x_0$.

4.1.3 Newell's "three-detector" problem



Newell's "three-detector" problem consists in predicting the traffic states at a location in between two loop detectors. Here the boundary data $G_U(t)$ and $G_D(t)$ is given at two locations x_U and x_D on a homogeneous roadway segment (as in Fig.) and one is interested in the cumulative count curve at a location $x, x_U < x < x_D$. In this case, the general recipe (4.3) simplifies to finding the minimum of two terms:

$$N_P = \min\{f(t_U), f(t_D)\}, \qquad \text{with:} \qquad (4.17a)$$

$$f(t_U) = G_U(t_U), \qquad (4.17b)$$

$$f(t_D) = G_D(t_D) + K(x_D - x), \qquad (4.17c)$$

$$t_U = t - (x - x_U)/u, \qquad (4.17d)$$

$$t_D = t - (x_D - x)/w, \qquad (4.17e)$$

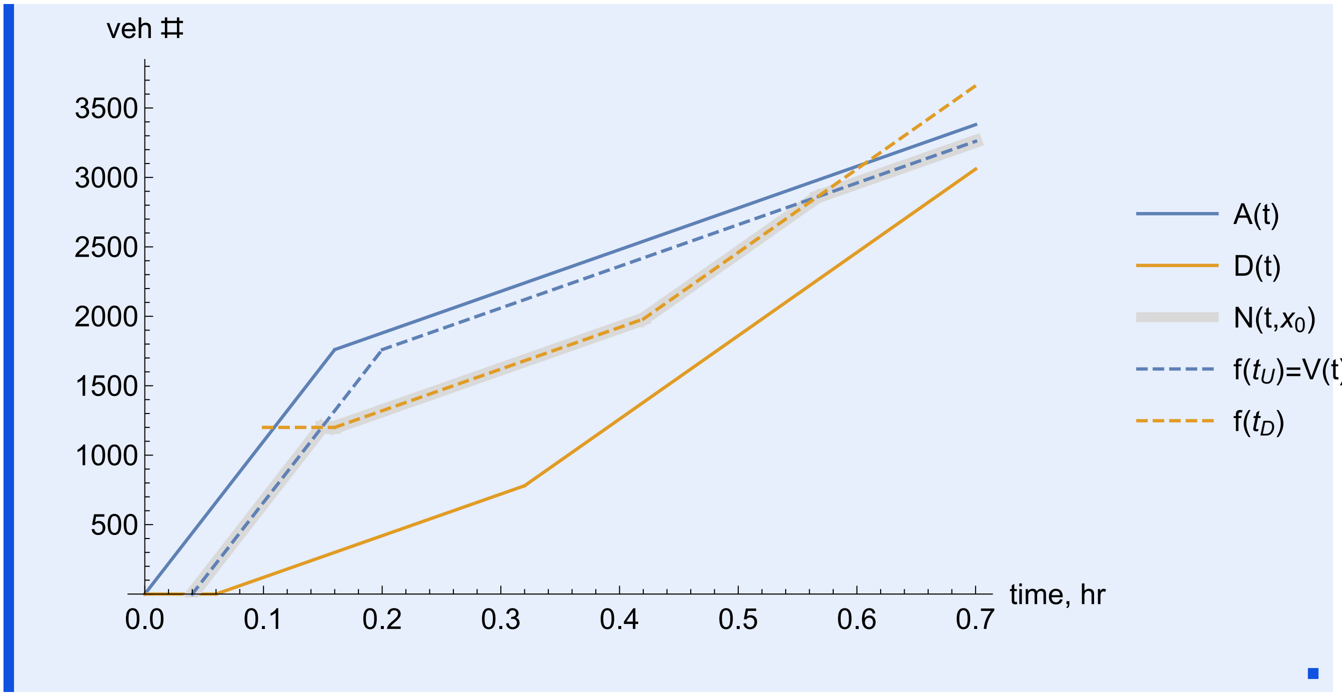
> **The trajectory of the shock** can be found by setting $f(t_U) = f(t_D)$.

Problem (4.17) is known as Newell's "three-detector method" and its solution can be represented graphically. To see this, notice that the term $G_U(t_U)$ represents the virtual arrivals at location x, as in chapter 3, while the term $G_D(t_D) + K(x_D - x)$ represents the possible departures at point P under congestion in the segment (x, x_D); it is obtained by shifting the departure curve $G_D(t)$ to the right by the wave travel time and up by the jam accumulation $K(x_D - x)$.

Example 4.3. — 3-detector problem On a 4-lane freeway you know the *flow* at $x = 0$ and at $x = 6$ km:

$$N_t(t,0) = \begin{cases} 11000 & t < 0.16 \\ 3000 & \text{otherwise} \end{cases} \quad \text{and} \quad N_t(t,4) = \begin{cases} 0 & t < 0.06 \\ 3000 & 0.06 < t < 0.32 \\ 6000 & \text{otherwise} \end{cases}$$

a) Find the N-curve at location $x_0 = 4$ km for $t < 1$ hr
b) At what times the back-of-the-queue crosses x_0 km while receding?

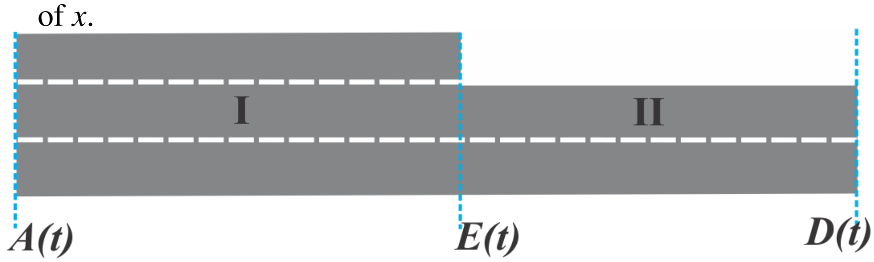


Inhomogeneous highways

If the highway can be partitioned into segments that are homogeneous, we can apply the three-detector method repeatedly as follows. For a highway with arrival and departure curves $A(t)$ and $D(t)$, that consists of two homogeneous segments I and II as in the figure, we would have:

Step 1. Find the N curve at point $E, E(t)$

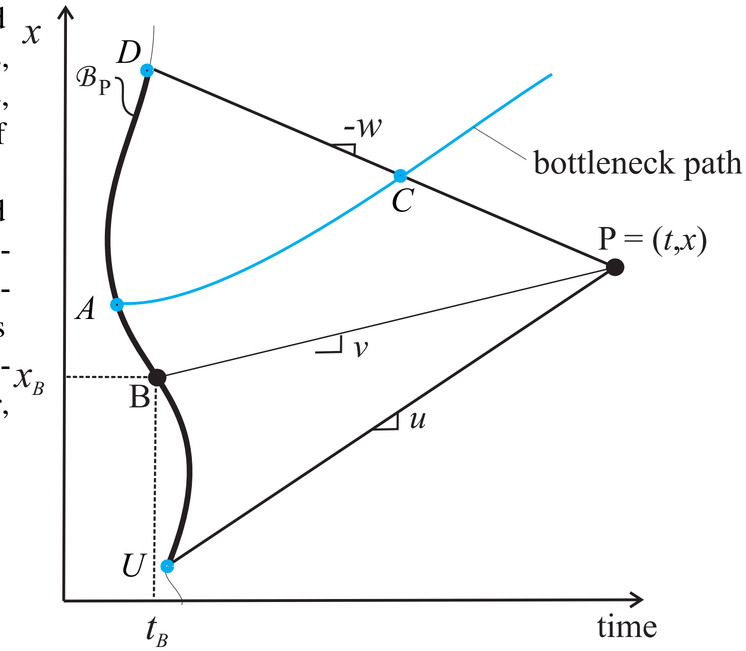
Step 2. Solve a 3-detector problem for point x using $E(t)$ and either $A(t)$ or $D(t)$, depending on the location of x.



To perform step 1 we first calculate the shifted arrivals at E, from $A(t)$ using u for segment I, u^I ; and then the possible departures using $D(t)$ and the shifts appropriate for segment II. Then, analysis of the shortest path problem shows that $E(t)$ is the highest curve that can be drawn beneath and with slope less than or equal to $\min\{Q^I, Q^{II}\}$.

4.1.4 Bottlenecks

Bottlenecks, fixed and moving, can be treated as "shortcuts" in the minimum path calculations, with costs given by its maximum passing rate, $R_B(v)$, which may be a function of the speed of the bottleneck.
In the figure this means that we need to add the path "P-C-A" to the problem without bottlenecks. The cost along this path corresponds to $(x - x_C)K$ along segments PC, plus $\int_{t_A}^{t_C} R(v(t))dt$ along CA. If the bottleneck passing rate is assumed independent its speed, say r, then this integral simplifies to $(t_C - t_A)r$.



4.2 Numerical solutions

Here we introduce exact numerical solution methods assuming a triangular fundamental diagram.

4.2.1 Variational networks

Daganzo [0] introduced time-space networks to solve the traffic problem using shortest paths. Each link i in these “variational networks" is defined by its: (i) slope v_i: wave speed, (ii) cost c_i: maximum number of vehicles that can pass, (iii) time length τ_i, and (iv) distance length $\delta_i = \tau_i v_i$. The cost to be used in each link becomes:

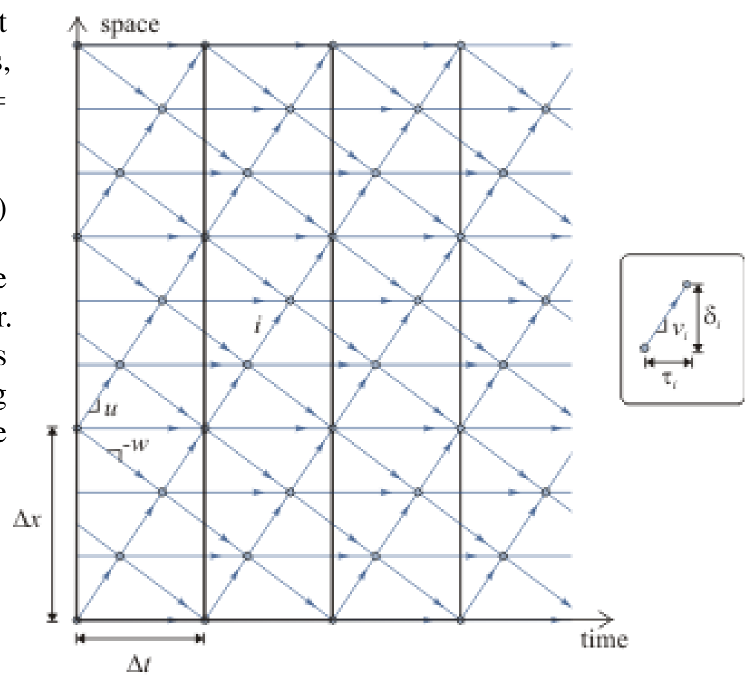
$$c_i = R(v_i)\tau_i. \tag{4.18}$$

The advantage of this method is that it is free of numerical errors but it may be cumbersome to work with unless θ is an integer. In that case, as illustrated in the Fig. for $\theta = \frac{u}{w} = 2$, the location of nodes align on a grid pattern. This allows defining a conventional grid with cell size $\Delta t, \Delta x$ where inflows may be assumed constant.

For each link i: (i) slope v_i: wave speed, (ii) cost c_i: maximum number of vehicles that can pass, (iii) time length τ_i, and (iv) distance length $\delta_i = \tau_i v_i$. The cost to be used in each link becomes:

$$c_i = R(v_i)\tau_i. \tag{4.19}$$

Method free of numerical errors but may be cumbersome to work with unless θ is an integer. In the Fig. $\theta = \frac{u}{w} = 2$,and the location of nodes align on a grid pattern. This allows defining a conventional grid with cell size $\Delta t, \Delta x$ where inflows may be assumed constant.



4.2.2 Time stepping methods

In this section the space and time dimensions are discretized in increments Δx and Δt. A “tilde" will denote a dimensionless quantity:

$$\widetilde{t} = t/\Delta t$$
$$\widetilde{x} = x/\Delta x.$$

These dimensionless quantities are restricted to assume integer values only, and we will carefully choose the increments so that rounding operations are not needed. A necessary condition to accomplish this is:

$$\theta = \frac{u}{w} \quad \text{is an integer,} \tag{4.21}$$

which should be close to 6 or 7 for typical freeways.

4.2.3 IVP models

The basic idea is to advance time in increments of Δt and at each time step find $N(t,x)$ by solving an IVP with initial data $N(t-\Delta t, x)$. If Δx is small enough so that the initial data can be assumed **linear between lattice points**, it follows from (4.14) that the set of candidates can only be lattice points $x_U, x_U+\Delta x, x_U+2\Delta x, \ldots x_D$. If one chooses $\Delta x = \delta\Delta n, \Delta t = \tau\Delta n$ and arbitrary Δn this set of candidates can be written as $x - i\Delta x, i = -1, 0, \ldots \theta$, and the solution can be written as:

$$N(t,x) = \min_{i=-1,0,\ldots\theta} \{N(t-\Delta t, x-i\Delta x) + \Delta t Q - i\Delta x K_c\}. \tag{4.22}$$

This can be rewritten as:

Lattice N-model In this model the coordinate system $(\tilde{t}, \tilde{x})$ is discrete and the dependent $N_{\tilde{t},\tilde{x}}$ is continuous.

$$N_{\tilde{t},\tilde{x}} = \min_{i=-1,0,\ldots\theta} \left\{ N_{\tilde{t}-1,\tilde{x}-i} + \Delta n \frac{\theta - i}{\theta + 1} \right\} \tag{4.23}$$

with discretization: $\Delta x = \delta\Delta n, \Delta t = \tau\Delta n$ and arbitrary Δn.

Taking $\Delta n = \theta + 1$ in (4.23) gives a Cellular automata (CA) model because it becomes an operator whose input and output are integer values.

CA N-model Here the function $N_{\tilde{t},\tilde{x}}$ is defined on the positive integers, and represents the vehicle number but in increments of $\theta + 1$.

$$N_{\tilde{t},\tilde{x}} = \min_{i=-1,0,\ldots\theta} \left\{ N_{\tilde{t}-1,\tilde{x}-i} + \theta - i \right\} \tag{4.24}$$

with discretization: $\Delta x = \delta\Delta n, \Delta t = \tau\Delta n$ and $\Delta n = \theta + 1$.

R For a simple spreadsheet implementation of these models can be found click here for $\theta = 2$; for a code in Mathematica click here. (Links will work only in the PDF version).

4.2.4 BVP models

Newell's three-detector method as a lattice implementation using $\Delta x = u\Delta t, x_U = x - \Delta x, x_D = x + \Delta x$:

$$N_{\tilde{t},\tilde{x}} = \min \left\{ N_{\tilde{t}-1,\tilde{x}-1}, N_{\tilde{t}-\theta,\tilde{x}+1} + K\Delta x \right\} \tag{4.25}$$

Notice that the term $\tilde{t} - \theta$ is inconvenient for practical applications with large networks because it implies the need for storing the state of the network for θ time-steps.

4.3 Problems

Unless otherwise indicated, assume that each lane of the facilities obeys a triangular fundamental diagram with $w = -20$ km/hr, $K = 150$ veh/km and $u = 100$ km/hr. Also, let Q and K_c be the resulting capacity and critical density, respectively. Make all the necessary assumptions, if needed.

Problem 4.1 — Shock path Derive $N(t,x)$ and the trajectory of the back-of-the-queue in the following cases:

a) as in Example (4.2) but with the following initial density:

$$k_0(x) = K_c(1+x), \qquad -1 \le x \le 5$$

b) a vehicle accelerating at a constant rate of 2 m/s^2 from $v = 40$ to $v = u$ on a 2-lane freeway with demand 4,000 vph.

Problem 4.2 — 3-detector problem On a 4-lane freeway you know the *flow* at $x = 0$ and at $x = 4$ km:

$$N_t(t,0) = \begin{cases} 7000 & t < 0.16 \\ 5000 & \text{otherwise} \end{cases} \quad \text{and} \quad N_t(t,4) = \begin{cases} 0 & t < 0.04 \\ 3000 & 0.04 < t < 0.32 \\ 6000 & \text{otherwise} \end{cases}$$

a) Find the N-curve at location $x = 2$ km for $t < 1$ hr

b) At what times the back-of-the-queue crosses $x = 2$ km while receding?

Problem 4.3 — Effect of initial data in the 3-detector problem Identify the cases when the initial data can affect the solution of the 3-detector problem seen in class.

Problem 4.4 — General boundary conditions Show that for general boundary data the first- and second-order optimality conditions are equivalent to the conditions for IVP plus BVP problems.

Problem 4.5 — 2 traffic lights* Two consecutive intersections that are 100 ft. apart on a one-way street are controlled by identically set, pre-timed traffic signals. Their cycle is one minute and the effective green phase for through traffic is 30 secs. Assuming no turning movements, determine:

a) the capacity of the system as a function of the offset,

b) the area in the time-space diagram upstream of intersection one such that if a red time on intersection two falls in that area then there is capacity loss,

c) (bonus) can you generalize a) for 3 traffic lights in series?

5. Car-following models

5.1 Introduction

The three representations of traffic flow In traffic flow there are three equivalent representations:

- $N(t,x)$: number of vehicles that have crossed location x by time t,
- $X(t,n)$: position of vehicle n at time t.
- $T(n,x)$: time vehicle n crosses x, and

All these representations correspond to the same surface in the three-dimensional space of vehicle number, time and distance, but expressed with respect to the three coordinate systems:

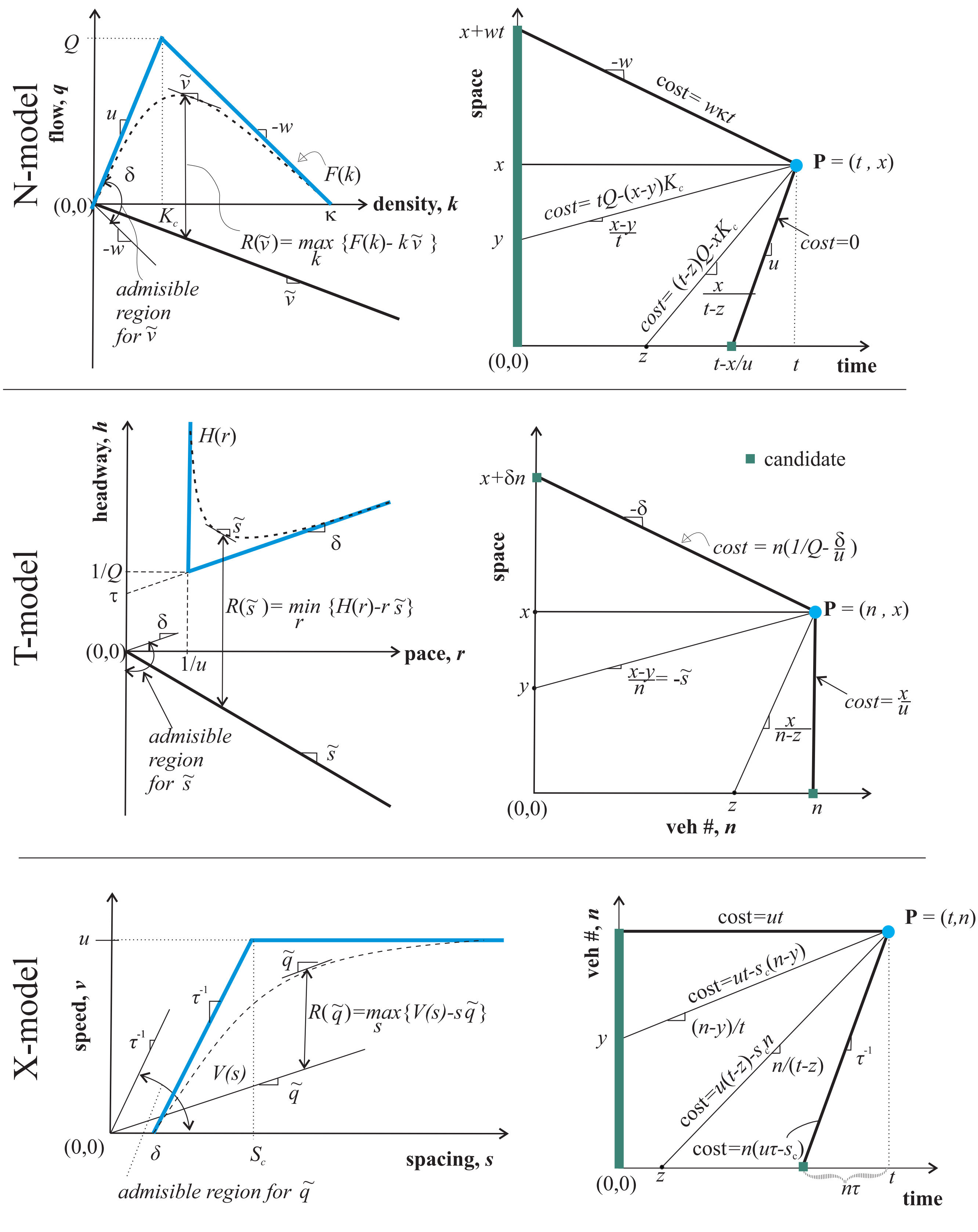
- Eulerian coordinates (t,x),
- Lagrangian coordinates (n,x), and (t,n) ← **C-F models**

	$N(t,x)$		$T(n,x)$		$X(t,n)$	
partials	N_t	$-N_x$	T_n	T_x	X_t	$-X_n$
symbol	$q(t,x)$	$k(t,x)$	$h(n,x)$	$r(n,x)$	$v(t,n)$	$s(t,n)$
name	flow	density	headway	pace	speed	spacing

Table 5.1: Coordinate systems and variables definition for the three representations

	$N(t,x)$	$T(n,x)$	$X(t,n)$
HJ-PDE	$q = F(k)$	$h = H(r)$	$v = V(s)$
Hamiltonian, FD	$F(k)$	$H(r)$	$V(s)$
FD'	$\tilde{v}$,wave speed	$\tilde{s}$, wave spacing	$\tilde{q}$, wave flow
Lagrangian $R(\cdot)$	$\max_k\{F(k) - k\tilde{v}\}$	$\min_r\{H(r) - r\tilde{s}\}$	$\max_s\{V(s) - s\tilde{q}\}$
$R(\cdot)$ with ΔFD	$Q - K_c\tilde{v}$	$1/Q - \tilde{s}/u$	$u - S_c\tilde{q}$

Table 5.2: Key elements of the Hamilton-Jacobi theory for the three coordinate systems.



N-model
flow, q
Q
u
δ
-w
F(k)
(0,0)
K_c
κ
density, k
R(ṽ)= max {F(k)- k ṽ }
k
admisible
region
for ṽ
ṽ
x+wt
-w
cost= wκt
space
x
P = (t , x)
cost= tQ-(x-y)K_c
y
x-y
t
cost=0
cost= (t-z)Q-xK_c
u
x
t-z
(0,0)
z
t-x/u
t
time
T-model
headway, h
H(r)
s̃
δ
1/Q
τ
R(s̃)= min {H(r)-r s̃}
r
(0,0)
1/u
pace, r
admisible
region
for s̃
candidate
x+δn
-δ
cost = n(1/Q- δ/u)
P = (n , x)
x-y
n
= -s̃
cost= x/u
x
n-z
(0,0)
z
n
veh #, n
X-model
u
speed, v
q̃
τ^-1
R(q̃)=max{V(s)-sq̃}
s
V(s)
(0,0)
δ
S_c
spacing, s
admisible region for q̃
veh #, n
cost=ut
P = (t,n)
cost=ut-s_c(n-y)
y
(n-y)/t
cost=u(t-z)-s_c n
n/(t-z)
cost=n(uτ-s_c)
(0,0)
z
nτ
t
time

From the theory of Hamilton-Jacobi PDEs seen in Chapter 4, the solution for the three models can be expressed as:

$$N(t,x) = \min_{B \in \mathscr{B}_P} \left\{ G(t_B, x_B) + (t - t_B) R\left(\frac{x - x_B}{t - t_B}\right) \right\} \tag{5.1a}$$

$$T(n,x) = \max_{B \in \mathscr{B}_P} \left\{ G(n_B, x_B) + (n - n_B) R\left(-\frac{x - x_B}{n - n_B}\right) \right\} \tag{5.1b}$$

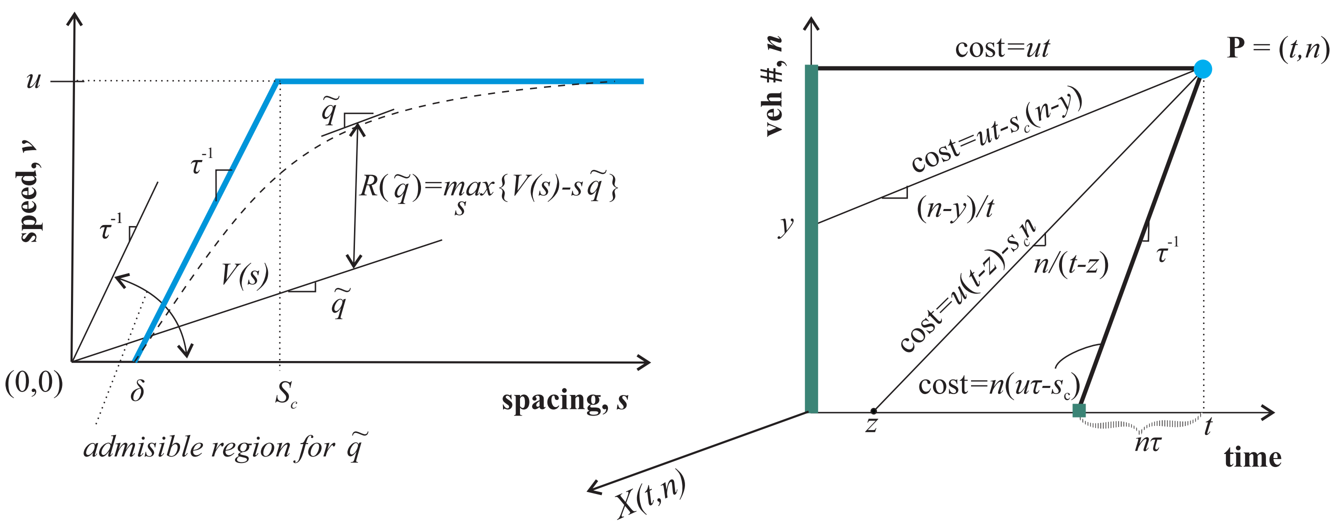
$$X(t,n) = \min_{B \in \mathscr{B}_P} \left\{ G(t_B, n_B) + (t - t_B) R\left(\frac{n - n_B}{t - t_B}\right) \right\} \tag{5.1c}$$

Interpretation of the data $G(\cdot)$ in IVPs and BVPs per model:

	IVP	BVP
$N(t,x)$	$N(0,x)$: cumulative vehicle profile at $t=0$	$N(t,0)$: cumulative count curve at $x=0$
$T(n,x)$	$T(0,x)$: trajectory of the lead vehicle	$T(n,0)$: time every vehicle enters the freeway
$X(t,n)$	$X(0,n)$: position of all vehicles at $t=0$	$X(t,0)$: trajectory of the lead vehicle

Notice that the BVP or IVP problems alone are not very meaningful for C-F models (T- and X-models); eg, a single vehicle trajectories cannot possibly provide information on the neighboring traffic conditions.

5.2 X-models



If the flow-density fundamental diagram is triangular, then

$$V(s) = \min\{u, s/\tau - w\} \quad \text{and} \quad R(\tilde{q}) = u - S_c \tilde{q}, \tag{5.2}$$

and the general car-following model (5.1c) becomes:

$$X(t,n) = \min \left\{ \min_{y \in \mathscr{B}_P} \{X(0,y) + ut - S_c(n-y)\}, X(t - n\tau, 0) - n\delta \right\} \tag{5.3}$$

after noting that $u\tau - S_c = -\delta$ in (5.3).

When the **initial data is linear** (constant spacing) the above recipe (5.3) involves the minimum of only two terms, i.e.:

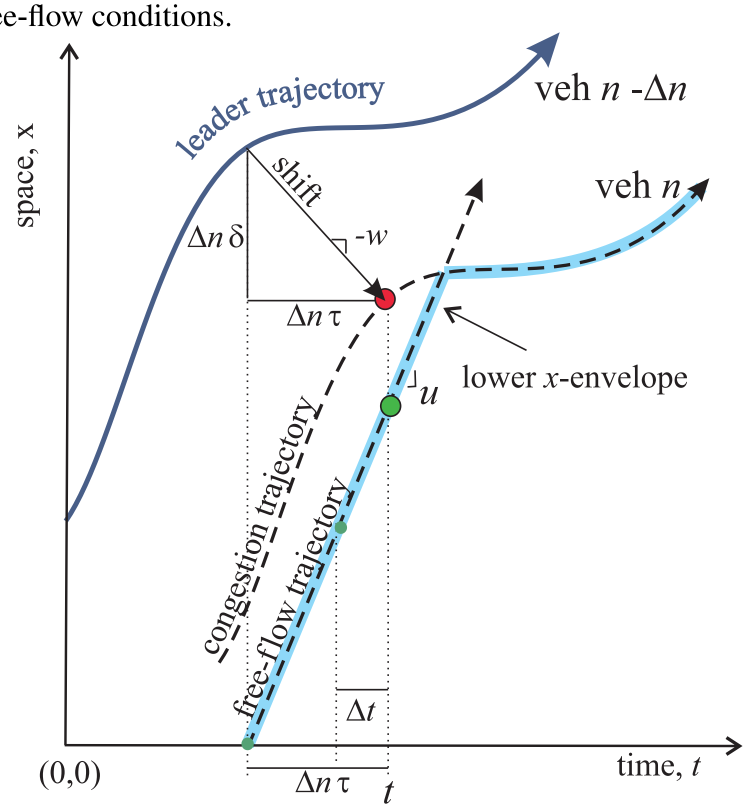
$$X(t,n) = \min\{X(0,n) + ut, X(t-n\tau,0) - n\delta\} \tag{5.4}$$

Linear initial data between grid points is a common assumption in numerical methods. The smaller the Δn, the better this assumption is.

Interpreting the origin $(0,0)$ as $(t-\Delta t, n-\Delta n)$ in the above figure, the recipe (5.4) becomes:

$$X(t,n) = \min\{X(t-\Delta t,n) + u\Delta t, X(t-\Delta n\tau, n-\Delta n) - \Delta n\delta\} \tag{5.5}$$

and has the following graphical interpretation: the trajectory of the subject vehicle is the lower envelope between its leader's trajectory shifted along the characteristic of slope $-w$, and its trajectory under free-flow conditions.



Newell's CF model. Set $\Delta n = 1$ in (5.5) to get:

$$X(t,n) = \min\{\underbrace{X(t-\tau,n) + u\tau}_{\text{free-flow term}}, \underbrace{X(t-\tau,n-1) - \delta}_{\text{congestion term}}\} \tag{5.6}$$

R An alternative derivation of Newell's CF model is to start with the fundamental diagram H-J PDE:

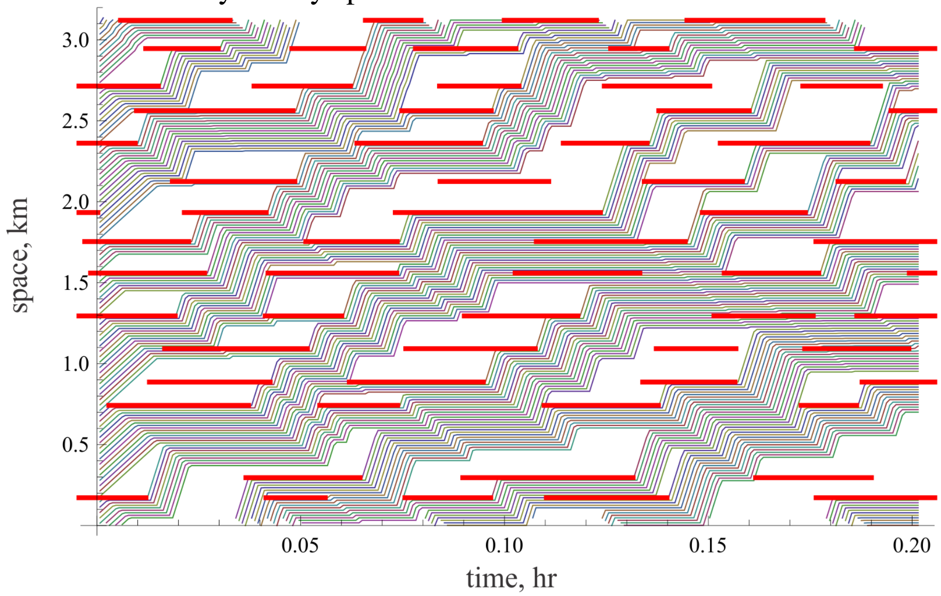
$$X_t - V(-X_n) = 0 \tag{5.7}$$

where $V(s)$ is the spacing-speed fundamental diagram. If the flow-density fundamental diagram is triangular, then using finite differences

$$X_t \approx (X(t+\Delta t,n) - X(t,n))/\Delta t,$$
$$X_n \approx (X(t,n+\Delta n) - X(t,n))/\Delta n$$

in combination with (5.2) and (5.7) gives (5.6) if $\Delta t = \tau$ and $\Delta n = 1$.

The figure below is an output of Newell's car following model for a ring road with 14 traffic lights, with vehicles initially evenly spaced.



5.2.1 Exact Discrete models

Similarly as for the N-model in the previous chapter, exact numerical solutions can be derived. We discretize in increments $\Delta n, \Delta t, \Delta x$. A "tilde" will denote a dimensionless quantity, e.g. $\widetilde{x} = x/\Delta x$ or $\widetilde{X}(\cdot) = X(\cdot)/\Delta x$. These dimensionless quantities are restricted to assume integer values only, and we will carefully choose the increments so that rounding operations are not needed. A necessary condition to accomplish this is:

$$\theta = \frac{u}{w} \quad \text{is an integer,} \tag{5.9}$$

which should be close to 6 or 7 for typical freeways.

Two types of discrete implementations are described here: (i) in a "lattice" implementation the coordinate system is discrete, e.g. $(\widetilde{t}, \widetilde{x})$, but the dependent variable is continuous, e.g. $X_{\widetilde{t},\widetilde{x}}$; (ii) in a cellular automata (CA) implementation everything is discrete, e.g. $\widetilde{X}_{\widetilde{t},\widetilde{x}}$. We use arguments in subscripts to emphasize that they are discrete variables.

We discretize in increments $\Delta n, \Delta t, \Delta x$. A "tilde" will denote a dimensionless quantity, e.g. $\widetilde{x} = x/\Delta x$ or $\widetilde{X}(\cdot) = X(\cdot)/\Delta x$. A necessary condition to accomplish exact methods is:

$$\theta = \frac{u}{w} \quad \text{is an integer,} \tag{5.10}$$

which should be close to 6 or 7 for typical freeways.

Two types of discrete implementations are described here: (i) in a "lattice" implementation the coordinate system is discrete, e.g. $(\widetilde{t}, \widetilde{x})$, but the dependent variable is continuous, e.g. $X_{\widetilde{t},\widetilde{x}}$; (ii) in a cellular automata (CA) implementation everything is discrete, e.g. $\widetilde{X}_{\widetilde{t},\widetilde{x}}$.

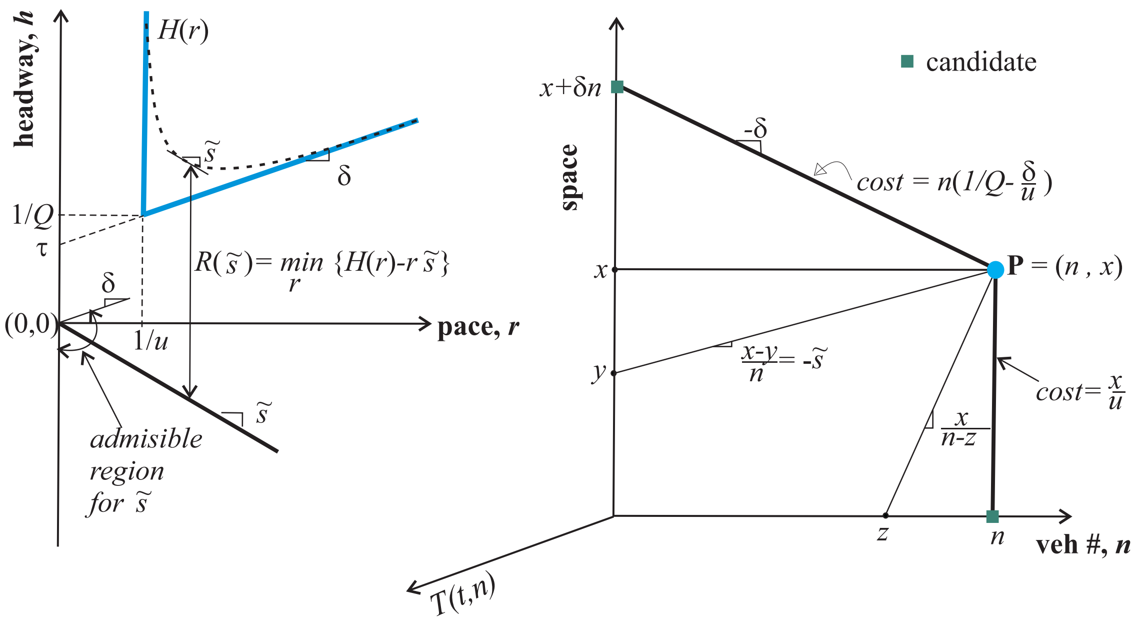
$$X_{\tilde{t},\tilde{n}} = \min\left\{X_{\tilde{t}-1,\tilde{n}} + u\Delta t, X_{\tilde{t}-1,\tilde{n}-1} - \Delta n\delta\right\}, \qquad \text{Lattice X-model} \qquad (5.11a)$$

$$\tilde{X}_{\tilde{t},\tilde{n}} = \min\left\{\tilde{X}_{\tilde{t}-1,\tilde{n}} + \theta, \tilde{X}_{\tilde{t}-1,\tilde{n}-1} - 1\right\}, \qquad \text{CA X-model} \qquad (5.11b)$$

with $\Delta t = \tau\Delta n, \Delta x = w\Delta t = \delta\Delta n$ and arbitrary Δn; $\theta = u/w$ (assumed integer).

The free parameters Δn can be interpreted as the size of the platoon. If $\Delta n = 1$ the model can be called a *microscopic* model, and if $\Delta n > 1$, *mesoscopic*.

5.3 T-models



General model:

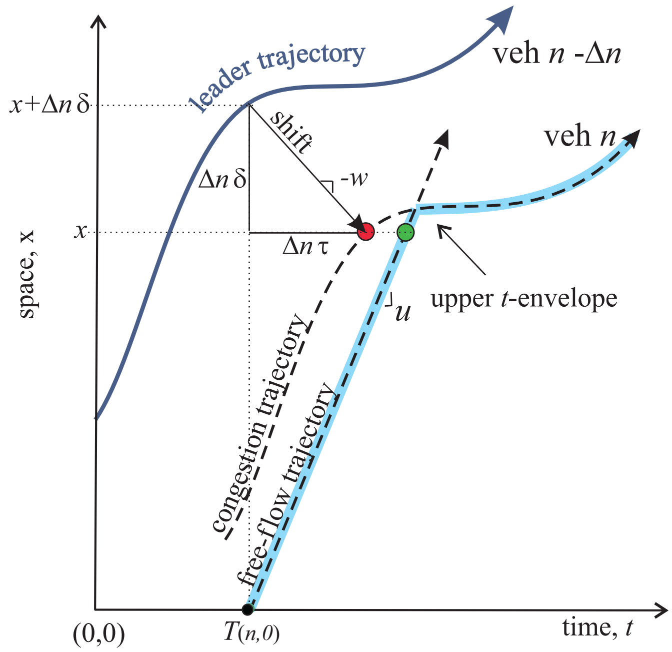
$$T(n,x) = \max_{B \in \mathscr{B}_P}\left\{G(n_B,x_B) + (n-n_B)R\left(-\frac{x-x_B}{n-n_B}\right)\right\} \tag{5.12}$$

If the flow-density fundamental diagram is triangular, then

$$R(\tilde{s}) = 1/Q - \tilde{s}/u, \tag{5.13}$$

and the costs in the figure above follow. Interpreting the origin in the figure as $(n-\Delta n, x-\Delta x)$ gives

$$T(n,x) = \max\{T(n,x-\Delta x) + \Delta x/u, T(n-\Delta n, x+\Delta n\delta) + \Delta n\tau\} \tag{5.14}$$



5.3.1 Exact Discrete models

$$T_{\tilde{n},\tilde{x}} = \max\left\{T_{\tilde{n},\tilde{x}-1} + \Delta x/u, T_{\tilde{n}-1,\tilde{x}+1} + \Delta x/w\right\}, \quad \text{Lattice T-model} \tag{5.15a}$$

$$\widetilde{T}_{\tilde{n},\tilde{x}} = \max\left\{\widetilde{T}_{\tilde{n},\tilde{x}-1} + 1, \widetilde{T}_{\tilde{n}-1,\tilde{x}+1} + \theta\right\}, \quad \text{CA T-model} \tag{5.15b}$$

with $\Delta x = \delta\Delta n, \Delta t = \Delta x/u = \delta\Delta n/u$ and arbitrary $\Delta n; \theta = u/w$ (assumed integer).

The lattice implementation of the T-model can be obtained dividing the arguments of $T(\cdot)$ by their respective increments. To ensure no numerical errors are introduced, the quantity $x + \Delta n\delta$ should be a lattice value, which can be achieved by choosing

$$\Delta x = \delta\Delta n \tag{5.16}$$

It follows that

$$T_{\tilde{n},\tilde{x}} = \max\left\{T_{\tilde{n},\tilde{x}-1} + \Delta x/u, T_{\tilde{n}-1,\tilde{x}+1} + \Delta x/w\right\} \tag{5.17}$$

is the lattice implementation of the T-model. Additionally, we can discretize time to obtain

$$\widetilde{T}_{\tilde{n},\tilde{x}} = \max\left\{\widetilde{T}_{\tilde{n},\tilde{x}-1} + \Delta x/(u\Delta t), \widetilde{T}_{\tilde{n}-1,\tilde{x}+1} + \Delta n\tau/\Delta t\right\} \tag{5.18}$$

This expression can be turned into a CA implementation by choosing:

$$\Delta t = \Delta x/u, \tag{5.19}$$

which gives the CA T-model:

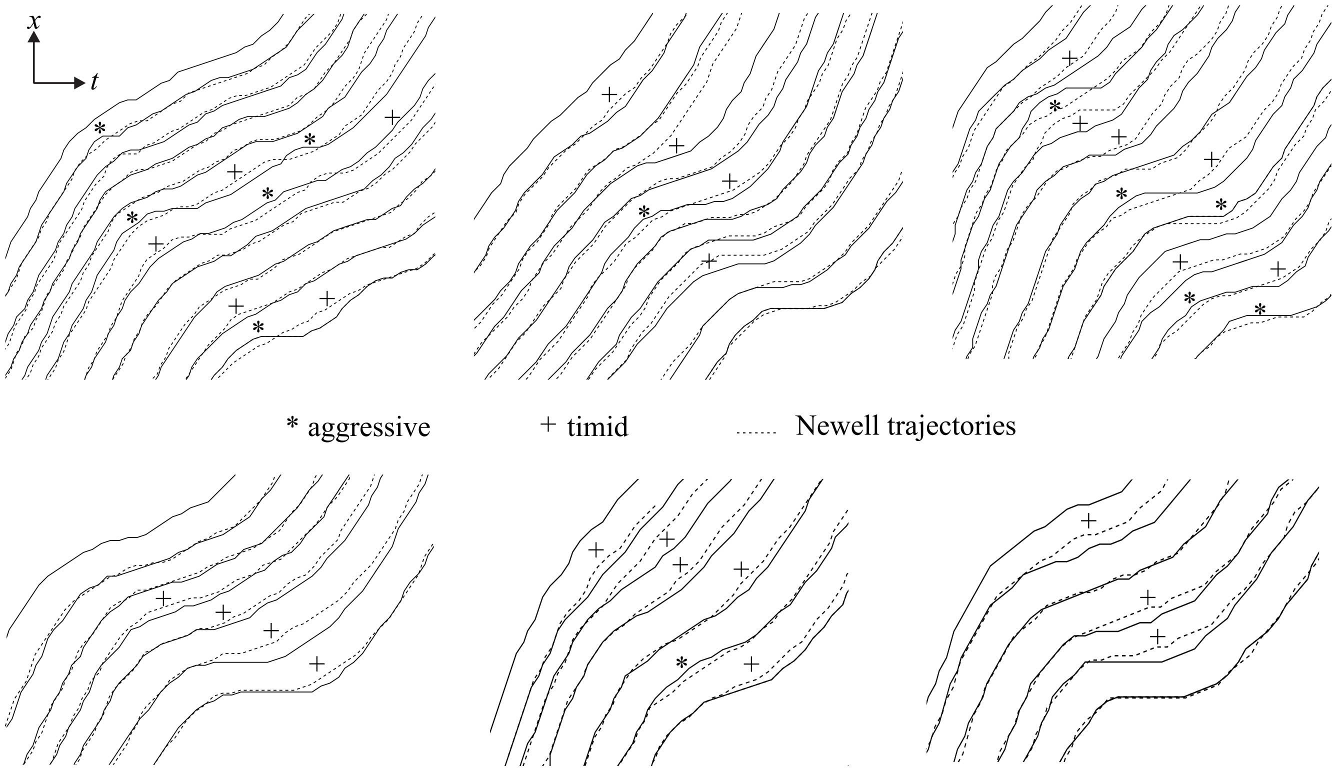
$$\widetilde{T}_{\tilde{n},\tilde{x}} = \max\left\{\widetilde{T}_{\tilde{n},\tilde{x}-1} + 1, \widetilde{T}_{\tilde{n}-1,\tilde{x}+1} + \theta\right\} \tag{5.20}$$

This relationship is a well-defined CA because it is an operation that takes integer values and returns an integer. Notice that both the lattice and CA expressions (5.15a) and (5.15b) are independent of the discretization size Δn as long as (5.16) and (5.19) are used. Also note that any sub-multiple of (5.16) such as $\Delta x = \delta \Delta n / c, c = 1, 2, \ldots$ also gives well-defined CA models:

$$\widetilde{T}_{\widetilde{n},\widetilde{x}} = \max\left\{\widetilde{T}_{\widetilde{n},\widetilde{x}-1} + 1, \widetilde{T}_{\widetilde{n}-1,\widetilde{x}+c} + c\theta\right\}, \tag{5.21}$$

which is convenient for its added flexibility. In particular, it allows us to use arbitrarily large Δn while keeping Δx reasonably small.

5.4 Deviations from Newell's model

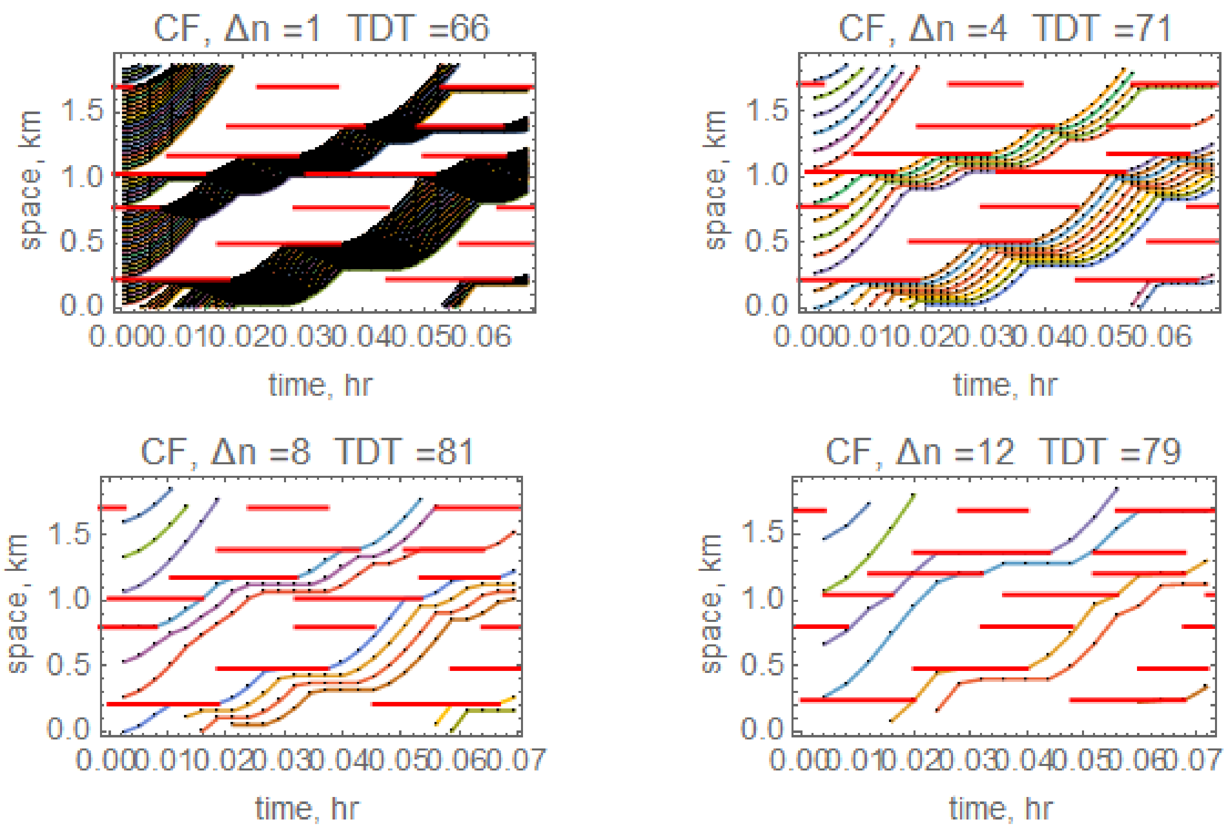


X-model with bounded accelerations We can remove infinite accelerations in the X-model (5.5) by incorporating a vehicle kinematics model, which were discussed in Chapter 1. Recall that these models give the desired vehicle acceleration, $a(v)$, when traveling at a speed v on an empty road.

If we let $\xi_n(t, v)$ be the resulting displacement at time t for vehicle n starting at a speed v at time 0, then all we need to do is replace the term $u\Delta t$ in the X-model by $\min\{u\Delta t, \xi_n(\tau, v_n(t-\Delta))\}$:

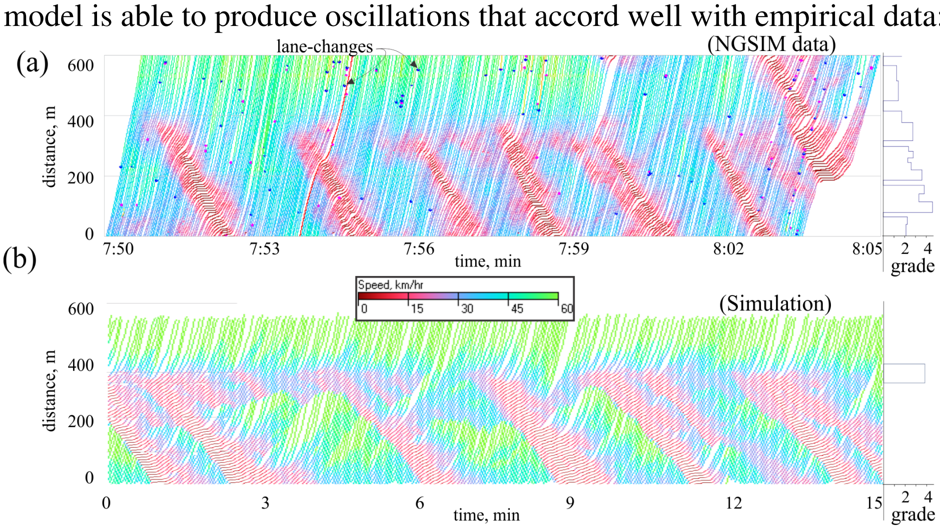
$$X(t,n) = \min\{\underbrace{X(t-\Delta t, n) + \min\{u\Delta t, \xi_n(\Delta t, v_n(t-\Delta t))}_{\text{free-flow term}}, \underbrace{X(t-\Delta n\tau, n-\Delta n) - \Delta n\delta}_{\text{congestion term}}\} \tag{5.22}$$

The figure below shows an output of the X-model with bounded accelerations on a signalized arterial, for different values of Δn.



5.4.1 Oscillations are due to human error

If the displacement ξ_n is assumed to be given by a linear acceleration model (chapter 1)plus a *white noise*, the model is able to produce oscillations that accord well with empirical data:



For more details see this paper (ppt).

5.5 Other car-following models

Some car-following models solved as a regular ODE, i.e. using numerical methods. There typically expressed in terms of the acceleration of vehicle j:

$$a_j(t) = f(v_j(t), \Delta x_j(t), \Delta v_j(t)) \tag{5.23}$$

where f is a stimulus function that depends on the speed of the current vehicle $v_j(t)$, spacing $\Delta x_j(t) = x_{j-1}(t) - x_j(t)$ and relative speed with the front vehicle $\Delta v_j(t) = v_{j-1}(t) - v_j(t)$.

Famous models:

1. **Optimal velocity model**: each driver tries to achieve an optimal velocity (fundamental diagram) based on the distance to the preceding vehicle and the speed difference between the vehicles.

$$a_j^t = [1/\tau][V^{opt}(\Delta x_j^t) - v_j^t] \tag{5.24}$$

   where $\frac{1}{\tau}$ is called as sensitivity coefficient.

2. **General motor's car following models.** The most general model has the form,

$$a_{j+1}^t = \left[\frac{\alpha_{l,m}(v_{j+1}^t)^m}{(x_j^t - x_{j+1}^t)^l}\right][v_j^t - v_{j+1}^t] \tag{5.25}$$

   where l is a distance headway exponent and can take values from +4 to -1, m is a speed exponent and can take values from -2 to +2, and α is a sensitivity coefficient.

3. Treiber's IDM model

$$v_j'(t) = a\left(1 - \left(\frac{v_j}{v_0}\right)^{\delta} - \left(\frac{s^*(v_j, \Delta v_j)}{s_j}\right)^2\right) \tag{5.26}$$

for vehicle j, with

$$\Delta v_j = v_j - v_{j-1} \tag{5.27}$$

$$s_j = x_{j-1} - x_j - l_{j-1} \tag{5.28}$$

$$s^*(v_j, \Delta v_j) = s_0 + v_j T + \frac{v_j \Delta v_j}{2\sqrt{ab}} \tag{5.29}$$

v_0, s_0, T, a, and b are model parameters which have the following meaning:

Variable	Description	Value
v_0	Desired velocity	30 m/s
T	Safe time headway	1.5 s
a	Maximum acceleration	0.73 m/s^2
b	Comfortable Deceleration	1.67 m/s^2
δ	Acceleration exponent	4
s_0	Minimum distance	2 m
l_j	Vehicle length	5 m

The IDM model is solved as a regular ODE, i.e. using Runge-Kutta methods.
Click for a Mathematica implementation of → the IDM model (vehicle-by-vehicle).

4. Gipp's model (Aimsun)

$$v_j(t+T) = \min\{V_{\text{free}}(v_j^t), V_{\text{cong}}(\Delta x_j^t, v_j^t, v_{j-1}^t)\} \tag{5.30}$$

for vehicle j, with

$$V_{\text{free}}(v) = 2.5aT\left(1-\frac{v}{u}\right)\left(\frac{v}{u}+0.025\right)^{0.5}+v \tag{5.31}$$

$$V_{\text{cong}}(\Delta x, v, v_0) = \sqrt{b^2\left(\theta+\frac{T}{2}\right)^2+b\left(\frac{v_0^2}{b_0}+2(\Delta x - s_0)-Tv\right)}-b\left(\theta+\frac{T}{2}\right) \tag{5.32}$$

Model parameters which have the following meaning:

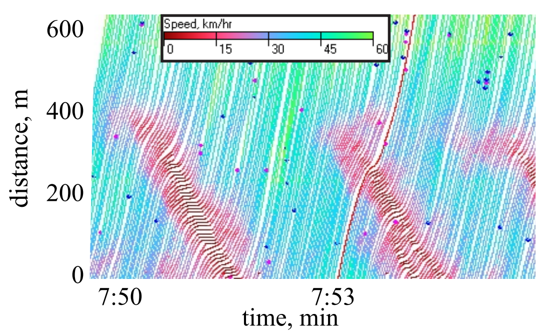
Variable	Description	Value
u	Desired velocity	30 m/s
T	Safe time headway	1.2 s
a	Maximum acceleration	1.7 m/s^2
b	Comfortable Deceleration	3 m/s^2
b_0	Emergency Deceleration	3.5 m/s^2
s_0	jam spacing	6.5 m
θ	reaction time	1/3 s

Gipp's model can be solved in two ways:

1. as a **delayed ODE** using numerical ODE methods,
2. as a **numerical method**, interpreting τ as a time step.

Click for a Mathematica implementation of → the Gipps model (vehicle-by-vehicle).

The problem: unrealistic stability behavior for reasonable parameter values.
PowerPoint presentation.



Definition: Types of instabilities. In traffic flow we have:

1. **String (aka Asymptotic) stability**: a disturbance grows in magnitude as it propagates through the platoon
2. Local instability: the situation in which a disturbance does not die out but rather increases with time
3. Metastability: traffic flow is stable w.r.t. disturbances with a small amplitude, but unstable w.r.t. larger disturbances

5.6 Problems

Problem 5.1 — A school bus Program the X-model numerical method of your choice to answer these questions. Suppose that the first vehicle on a single lane road is a school bus, and that its trajectory is:

$$x_0(t) = 50\,t + 0.12(1 + \sin(400\,t)) \tag{5.33}$$

where t is in hours and x in kilometers. Repeat 1-4 below for $\Delta n = 1$ and $\Delta n = 5$:

a) Plot the trajectory of 60 vehicles upstream of the school bus assuming that at t=0 the spacing between vehicles is $s_0 = 100$ m. Print this plot up to $t = 6$ min.
b) On the previous plot, use your judgment to draw with pencil and ruler the straight line that "best fits" the shock you observe. Compute the slope of this line.
c) Calculate the speed of the shock that you would obtain with kinematic wave theory assuming that the school bus travels at a constant speed v, where v is the average speed from (5.33). Compare with part b) and comment.
d) Using your simulation program, estimate the spacing s_0 such that the last vehicle reaches the back of the queue approximately at t = 6 min.

Problem 5.2 — Traffic light tailgating A recent empirical study by Virginia Tech researchers revealed that the discharge rate from a traffic light *is roughly independent* of the spacing of cars waiting in line during the red phase. Review this news article and related references and assess the research results in light of the concepts and models seen in class. As a starting point:

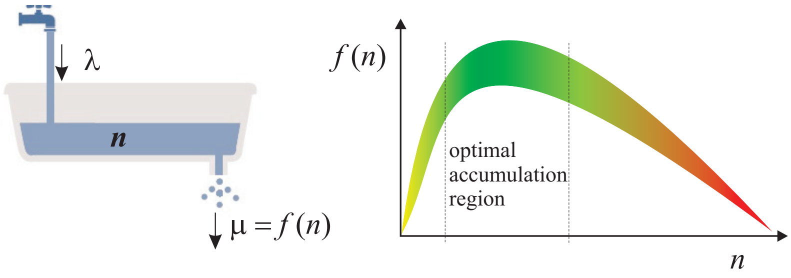
a) Review the corresponding Journal article critically.
b) Show the kinematic wave theory solution to this problem and see if it reproduces the empirical findings.
c) Use Newell's model with bounded acceleration to simulate the experiment and see if for some parameter combinations the empirical findings can be reproduced. (Don't forget to analyze the effects of the grade.)
d) Discuss your results.

6. Macroscopic models for cities

> Carlos Daganzo's presentation slides and video.

The recent discovery of the existence of a network-level Macroscopic Fundamental Diagram (MFD) on congested urban areas opened up a new paradigm. The MFD gives the average traffic state on a network as a function of the number of vehicles inside this network, roughly independently of trip origins and destinations, and route choice. This makes the MFD an invaluable tool to overcome the difficulties of traditional planning models.

Reservoir (aka bathtub) models for urban areas are the simplest ones to incorporate the effects of congestion. They are based on the simple principle that the variation in the amount of particles inside a reservoir is given by its inflow minus its outflow, and that the outflow is a known function of the accumulation. This function is the MFD, and appears to have strong empirical support in favor of a rather stable shape , and many control applications have been proposed since.

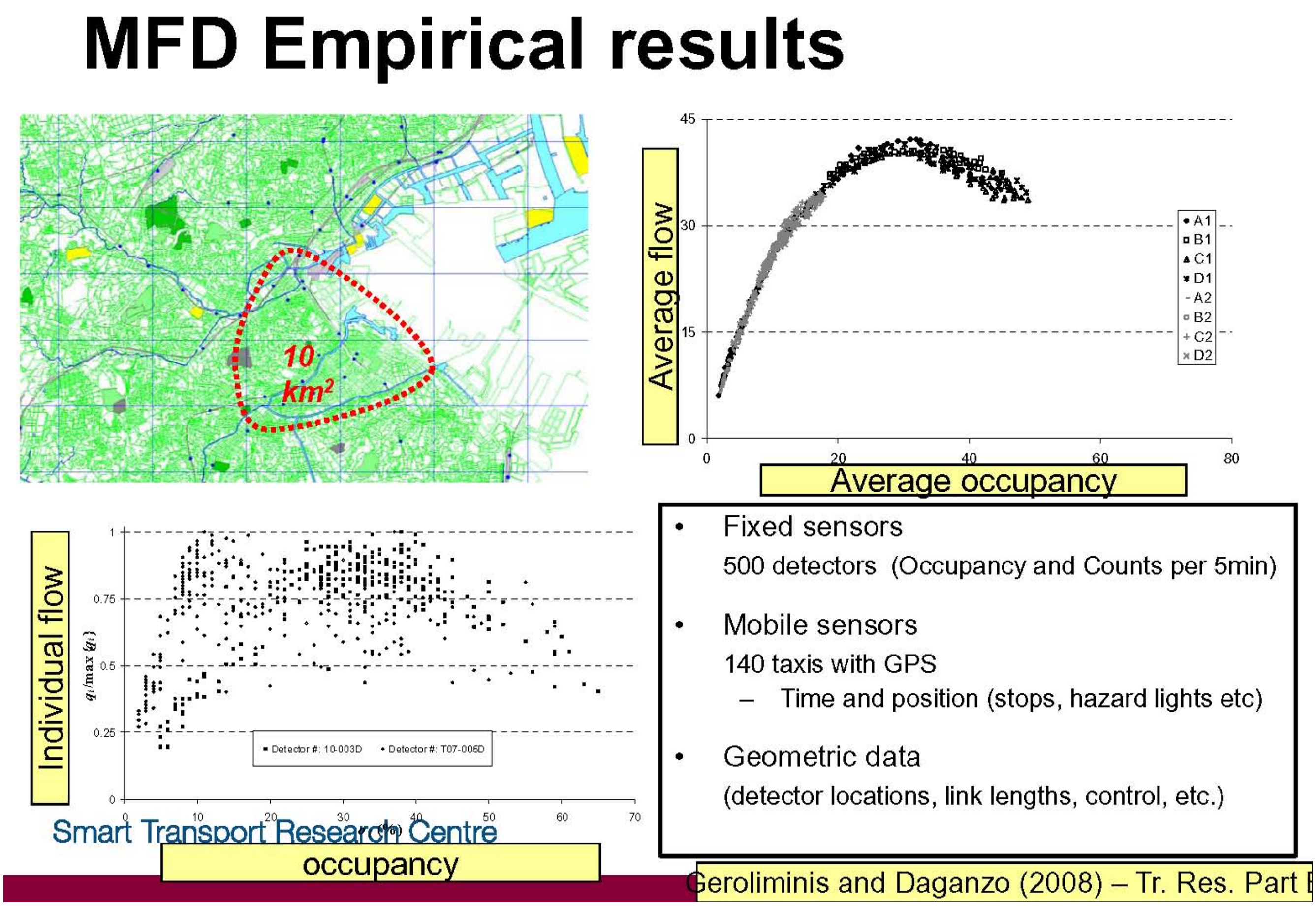


Main assumptions of a bathtub model:

1. the shape of the MFD

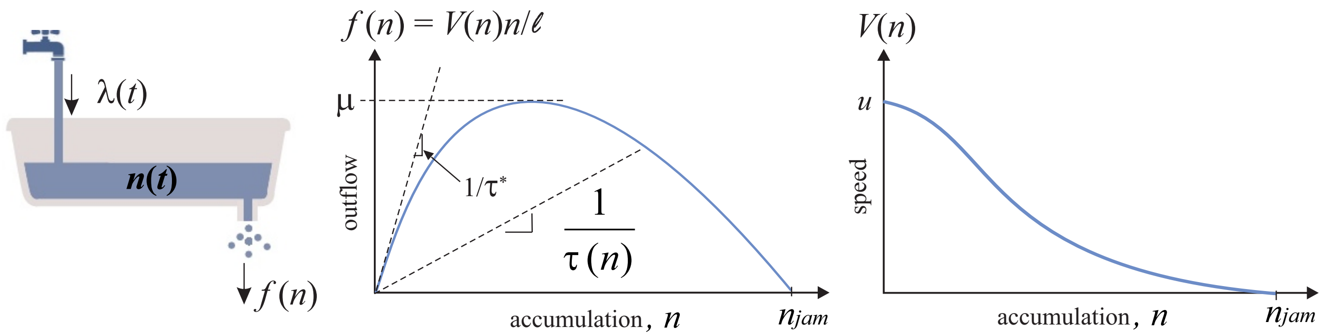
2. congestion is homogeneously distributed across the network, no hot spots
3. the travel time of a vehicle entering at time t is assumed a function of the accumulation at the *same* time, which makes it applicable only *when the inflow varies slowly*

6.1 The Yokohama MFD



6.2 The reservoir model

PowerPoint presentation / Paper



Definition: Link Notation. We will use the following notation:

- n_i = accumulation, the number of vehicles on link i (#)
- $\phi_i(t)$ = trip completion rate on link i (veh/km-hr)
- L_i = length of link i (km)
- K = jam density for one lane, assumed identical for all lanes in the network

Definition: Network Notation. We will use the following notation:

- $n \equiv \sum n_i$ = number of vehicles on the network (#)
- $f(n)$ = network outflow MFD = number of for trip completions per hr
- $\mu \equiv \max_n f(n)$ = capacity of the o-MFD (veh/hr)
- $V(n)$ = network speed MFD
- $Q(n)$ = network (average) and aflow MFD
- n_{jam} = network jam accumulation (#)
- $L \equiv \sum L_i$ = total network length (km)
- ℓ = trip length (km)
- $\tau(n) \equiv \ell/V(n)$ = the network travel time (hr)
- $\tau^* \equiv \ell/u$ = free-flow travel time in the network (hr) network
- $\lambda(t)$ = demand to flow into the network at time t (veh/hr)

The outflow from the network at link i, f_i is the number of for completions per hr on link i, $f_i = \phi_i L_i$. Combining this with the main MFD assumption $\phi_i = q_i/\ell$ gives

$$f_i = q_i L_i/\ell \tag{6.1}$$

The total outflow is then

$$\begin{aligned} f &= \sum f_i && (6.2a) \\ &= \sum q_i L_i/\ell && (6.2b) \\ &= \sum v_i n_i/\ell && (6.2c) \end{aligned}$$

where we used $q_i = v_i d_i$ and $n_i = L_i d_i$, (d_i = density) before the last quality. If the network is homogeneously congested then the traffic state on all links is equal to its average, $q_i \approx \overline{q}$ and $v_i \approx \overline{v}$ for all i, and from (6.2) it follows that

$$\begin{aligned} f &\approx \overline{q}\, L/\ell && (6.3a) \\ &\approx \overline{v}\, n/\ell && (6.3b) \end{aligned}$$

Given the existence of the MFD, we can say $\overline{q} = Q(n)$ and $\overline{v} = V(n)$, and we have:

Definition: The outflow-MFD The o-MFD gives the number of trip completions per unit time on the network, as a function of the number of vehicles in the network, n:

$$f(n) = Q(n)\,L/\ell \tag{6.4a}$$
$$= V(n)n/\ell \tag{6.4b}$$

The conservation equation in the reservoir model is given by the following ordinary differential equation (ODE):

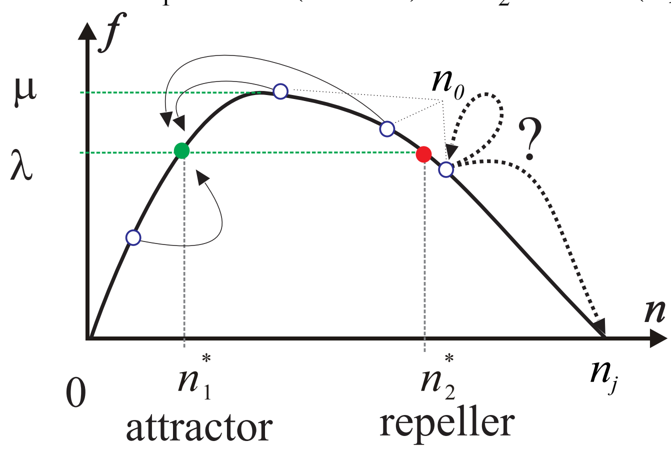
$$\textbf{n-ODE:} \begin{cases} n'(t) = \lambda(t) - f(n), & \text{(reservoir dynamics)} \quad (6.5a) \\ n(0) = n_0, & \text{(initial conditions)} \quad (6.5b) \end{cases}$$

where primes denote differentiation and n_0 specify the initial conditions.

Solution for the autonomous system

When demand is constant ($\lambda(t)$=λ) then (6.5) is said to be an *autonomous system* since its evolution depends only on the accumulation (and not on other time-dependent functions). Despite their simplicity, autonomous systems capture most of the dynamics under more general demand patterns in our case.

The solution of these systems is fully characterized by two critical accumulation that are the roots of $\lambda - f(n) = 0$: one in free-flow, n_1^*, and another in congestion, n_2^*.Since $f'(n_1^*) > 0$ and $f'(n_2^*) < 0$ it can be shown that n_1^* is stable (attractor) and n_2^* unstable (repellor).



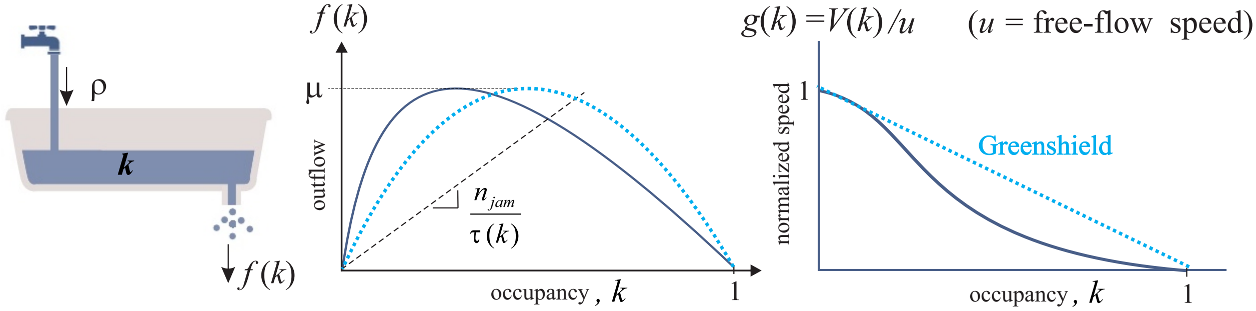
The "**?**" in the figure above refers to the assumption that one makes when the demand $\lambda(t)$ exceeds the outflow $f(n)$. A standard assumption in the literature is that demand should be restricted by the congested branch of the o-MFD, also called the supply function, $S(n)$:

$$\lambda(t) \leqslant S(n(t)). \quad \text{(supply constraint)} \tag{6.6}$$

to reflect that, in congestion, the inflow cannot exceed the outflow. This is consistent with the idea that all links in the region, including those on its perimeter, have the same (congested) flow and therefore if demand exceeds it, then a queue accumulates outside the region. In such cases, the accumulation remains constant; otherwise, the system tends to gridlock.

6.2.1 Analytical solutions

> For more details see the paper: Minimal parameter formulations of the dynamic user equilibrium using macroscop



It is convenient to express system dynamics in terms of the dimensionless variables occupancy, $k(t)$, and demand intensity $\rho(t)$:

$$k(t) \equiv n(t)/n_{jam}, \qquad 0 \leq k(t) \leq 1, \qquad \text{(occupancy)} \qquad (6.7a)$$

$$\rho(t) \equiv \lambda(t)/\mu, \qquad \text{(demand intensity)} \qquad (6.7b)$$

We assume a general speed-occupancy MFD, $V(k)$, of the form:

$$V(k) = ug(k), \qquad \text{(MFD speed-occupancy)} \qquad (6.8)$$

where u is the MFD free-flow speed (which is $\leq$ FD free-flow speed as it includes the effects of signals) and $g(k)$ is an arbitrary function of the occupancy satisfying $g(0) = 1, g(1) = 0$ and $g' < 0$.

If we measure time in units of $\tau^*, t = t/\tau^*$, it can be shown that the conservation ODE becomes:

$$\textbf{k-ODE:} \begin{cases} k'(t) = c\rho(t) - kg(k), & \text{(reservoir dynamics)} \quad (6.9a) \\ k(0) = k_0, & \text{(initial conditions)} \quad (6.9b) \end{cases}$$

where $c \equiv \mu\tau^*/n_j$ is the MFD "shape" parameter; $c = 1/4$ for parabolic (Greenshields) and $c = 1/2$ for isosceles $f(k)$.

Autonomous case solution:

$$k(t) = T^{-1}(T(n_0) + t), \qquad \text{(with } T(k) = \int dk/(c\rho - kg(k)).\text{)} \qquad (6.10)$$

For the Greenshields approximation, $g(k) \equiv 1 - k$, $c = 1/4$ and ρ is the only parameter! It can be shown that in our case:

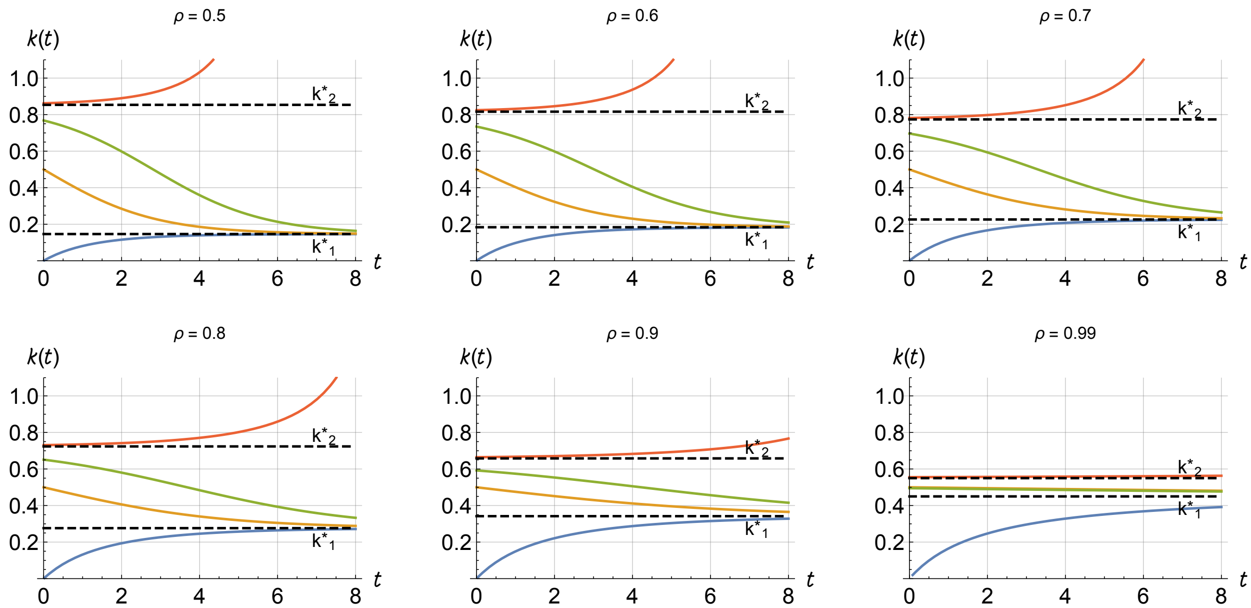
$$T(k) = \frac{1}{c_1} \tanh^{-1}\left(\frac{1/2 - k}{c_1}\right), \qquad (6.11)$$

with $c_1 = \sqrt{1-\rho}/2$, and thus the solution (6.10) can be expressed as:

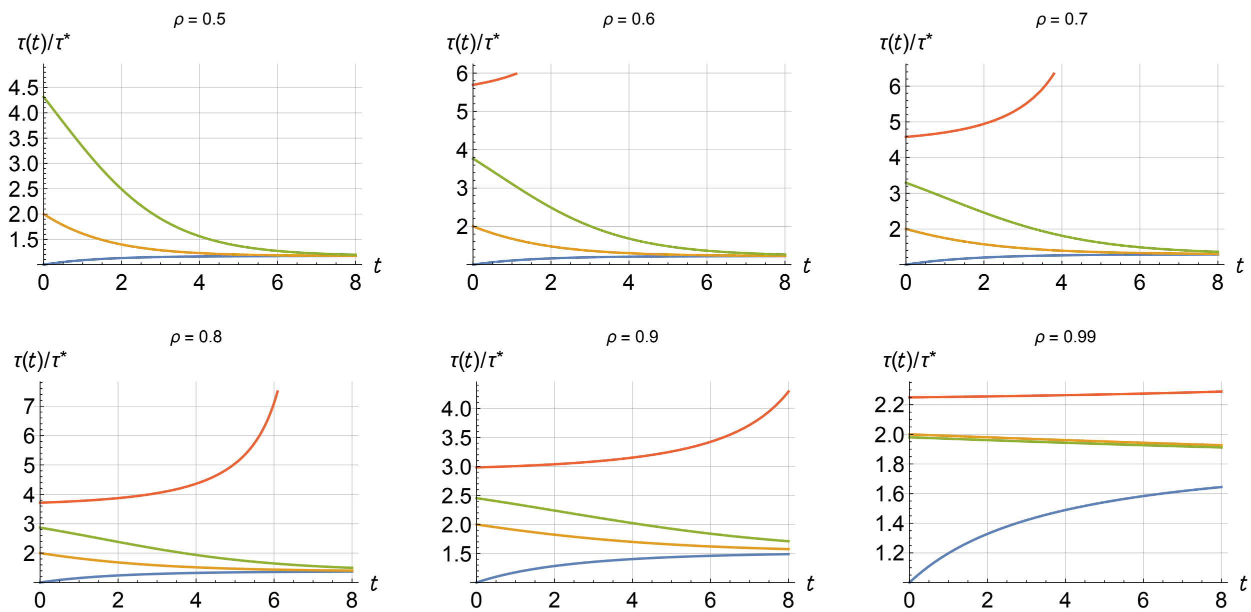
$$k(t) = 1/2 - c_1 \tanh(T(k_0) + c_1 t). \qquad \text{(autonomous, Greenshields solution)} \qquad (6.12)$$

The figure below shows the occupancy $k(t)$ for 4 initial values $k_0 = k(0)$ and 2 values for ρ. We can see that (i) when $\rho \leq 1$ then $k \to k_1^*$ if $k_0 \leq k_2^*$ and to 1 (gridlock) if $k_0 > k_2^*$, as expected, (ii) the convergence to gridlock happens at an increasing rate, (iii) the relaxation time of the system is comparable to $5\tau^*$ in most cases, (iv) when $\rho > 1$, i.e. when the demand exceeds the MFD capacity, the system converges to gridlock for all initial occupancies, as expected because the inflow is unrestricted.

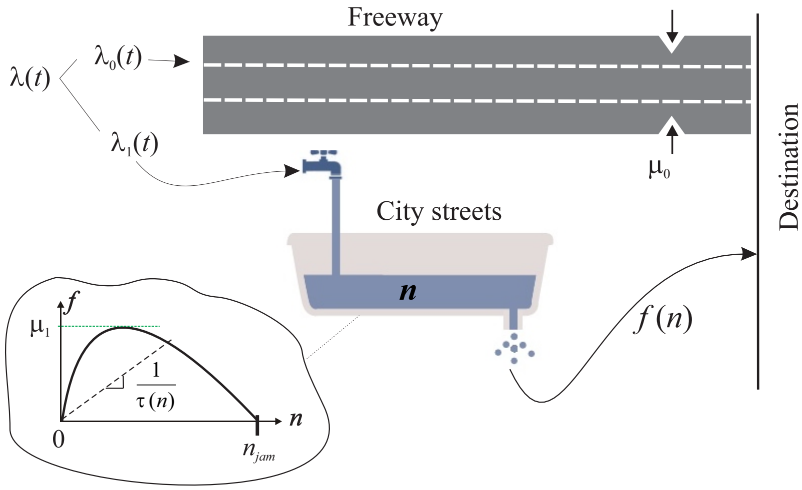
The charts below show the analytical solution (6.12) for several values of ρ. Recall that t is measured in units of τ^*.



The charts below show the corresponding travel time as a function of time, both measured in units of τ^*.



6.3 Freeway bottleneck vs city streets MFD



System dynamics:

$$\tau_0(t) = \tau_0^* + w_0(t), \qquad \text{(freeway travel time)} \qquad (6.13a)$$
$$\tau_1(t) = n(t)/f(n(t)), \qquad \text{(CS travel time)} \qquad (6.13b)$$
$$n'(t) = \lambda_1(t) - f(n(t)), \qquad \text{(reservoir dynamics)} \qquad (6.13c)$$
$$\lambda(t) = \lambda_0(t) + \lambda_1(t), \qquad \text{(demand conservation)} \qquad (6.13d)$$

and $w_0(t)$ is the freeway queuing delay, which is given by $w_0(t) = A_0(t)/\mu_0 - t \geq 0$; see Chapter 2.
Set $t = 0$ when $\tau_0(t) = \tau_1(t)$ and use UE condition in differential form:

$$\tau_0'(t) = \tau_1'(t)$$

and express system dynamics in terms of the occupancy k:

$$\begin{cases} k'(t) &= (c\rho(t) - kg(k))/(1 - cmg'/g^2), \\ k(0) &= k_0, \end{cases}$$

where :

$$m = \mu_0/\mu_1, \qquad \text{(capacity ratio)} \qquad (6.14a)$$
$$\rho(t) \equiv (\lambda(t) - \mu_0)/\mu_1 \qquad \text{(MFD demand intensity)} \qquad (6.14b)$$

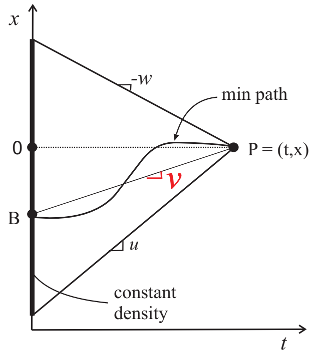
The solution is again $k(t) = T^{-1}(T(n_0) + t)$, with:

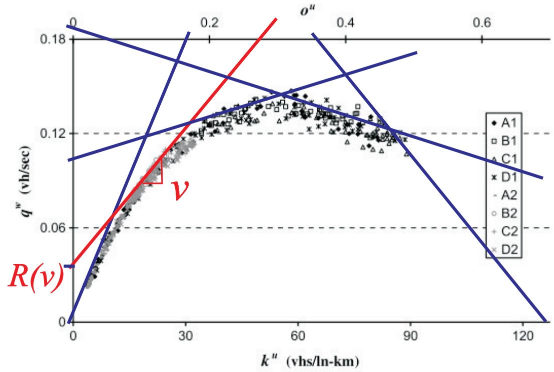
$$T(k) = \left(1 + (2-\rho)m/\rho^2\right) T^1(k) + \frac{m}{\rho^2}\left(\frac{\rho}{1-k} + 2\log\left(\frac{\rho - 4(1-k)k}{(1-k)^2}\right)\right) \qquad (6.15)$$

where $T^1(k)$ is the corresponding T-function for the single MFD problem (6.11).

- as $m \to 0$ (no freeway) then $T \to T^1$
- as $m \to \infty$ (no CS) then $T \to \infty$

6.4 MFD estimation: Method of Cuts





The method of cuts (MOC) is derived from the variational theory (Chapter 4), where we know that:

$$N_P = \min_B \{N_B + \Delta_{BP}\}, \tag{6.16}$$

where Δ_{BP} is the cost of the *minimum path* connecting boundary point $B = (t_B, x_B)$ and point P. For an initial value problem with a constant **density** k, $N_B = N_O + (x - x_B)k$ so:

$$N_P - N_O = \min_B \{(x - x_B)k + \Delta_{BP}\}, \tag{6.17}$$

The average flow MFD is defined as the steady-state flow at any location x; i.e.:

$$Q(k) \equiv \lim_{t\to\infty} \frac{1}{t}(N_P - N_O) \tag{6.18a}$$

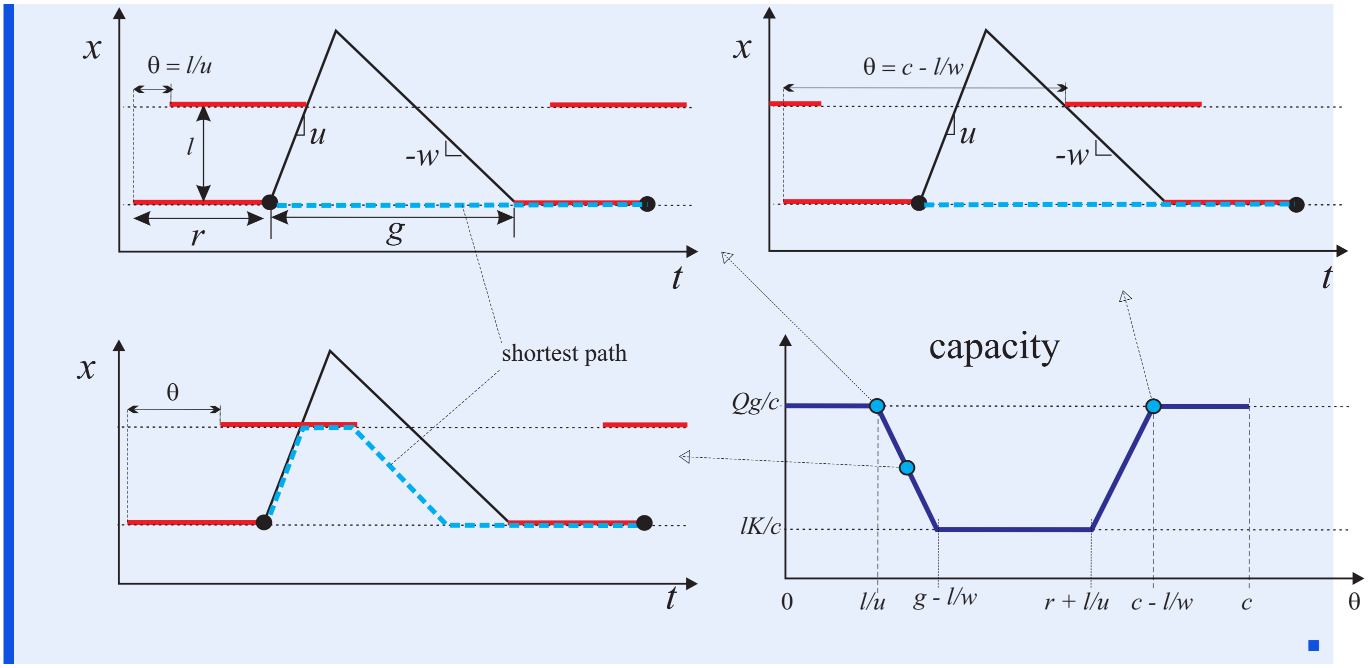
$$= \min_B \{\lim_{t\to\infty} \frac{1}{t}((x - x_B)k + \Delta_{BP})\}, \tag{6.18b}$$

$$= \min_v \{vk + R(v)\} \quad \text{(method of cuts).} \tag{6.18c}$$

Expression (6.18c) is the MOC (paper), where $v \equiv (x - x_B)/t$ is the average speed of the cut and $R(v) \equiv \Delta_{BP}/t$ is its maximum passing rate. Very useful for deterministic homogeneous problems where $R(v)$ can be evaluated analytically.

Definition: Homogeneous corridor: Sequence of street segments of identical lengths delimited by traffic signals with identical settings.

Example 6.1. — Capacity of 2 closely spaced intersections Two consecutive intersections that are 100 ft. apart on a one-way street are controlled by identically set, pre-timed traffic signals. Their cycle is one minute and the effective green phase for through traffic is 30 secs. Assuming no turning movements, determine the capacity of the system as a function of the offset.



6.4.1 Stochastic Method of Cuts

PowerPoint presentation / Paper

The MFD of urban networks :

a) mostly depends on 2 parameters

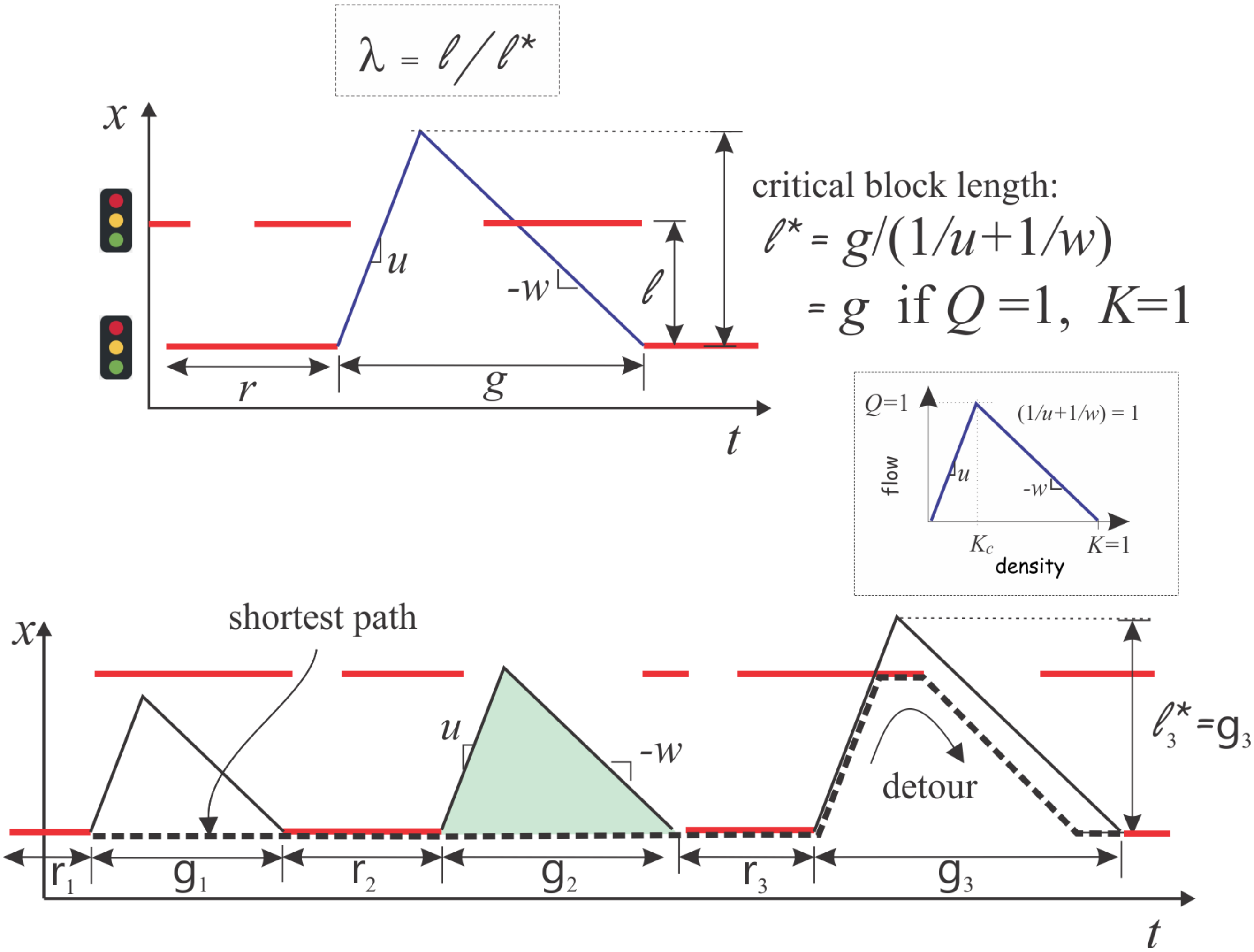
$$\lambda = \frac{E(\text{distance between traffic lights})}{E(\text{green time})}, \tag{6.19a}$$

$$\delta = COV(\text{distance between traffic lights}) \quad \text{and} \tag{6.19b}$$

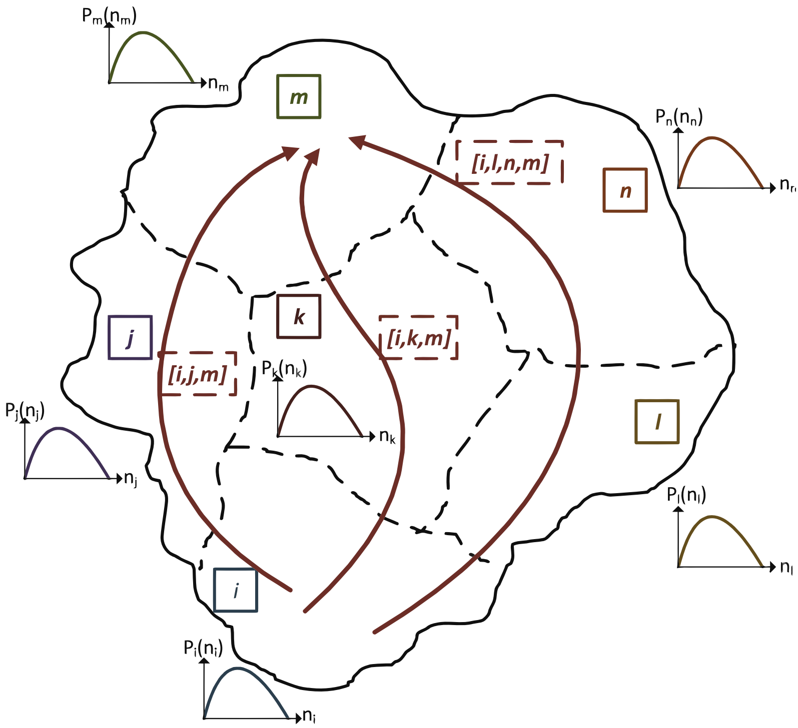
$$\rho = \frac{E(\text{red time})}{E(\text{green time})} \quad (= 1 \text{ for whole network}) \tag{6.19c}$$

b) is symmetric

Limitations: short blocks (λ<1)


λ = ℓ/ℓ*
x
u
-w
ℓ
r
g
t
critical block length:
ℓ* = g/(1/u+1/w)
= g if Q=1, K=1
Q=1
(1/u+1/w) = 1
flow
u
-w
Kc
K=1
density
shortest path
x
u
-w
detour
ℓ*3 =g3
r1
g1
r2
g2
r3
g3
t

6.5 Discrete space models



Various paths from region *i* to region *m* [0].

6.6 Continuum space models

Continuum pedestrian models and continuum-space DTA models are mathematically similar. The first formulation of this type of models was proposed in the context of pedestrian dynamics by **Hughes[0]**.

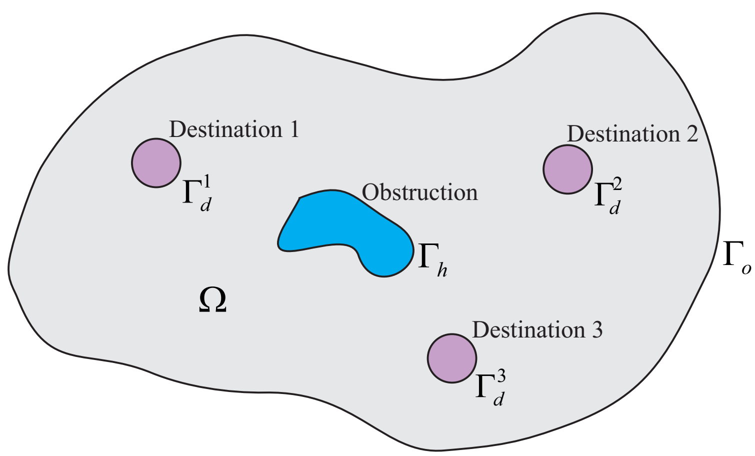


Figure 6.1: Illustration of continuum space with three destination areas and one obstruction .

The flow streamlines gives the trajectory of travelers. In fluid dynamics, streamlines are lines whose tangent at any point is in the direction of the velocity at that point. In particular, they are perpendicular to the speed potential.

Definition: Notation. We will use the following notation:

- density of travelers: $\rho(x,y,t)$ at location (x,y) at time t,
- velocity vector: $\mathbf{u}= (u_1(x,y,t),\ u_2(x,y,t))$
- flux vector: $\mathbf{f}(x,y,t) \equiv \rho\mathbf{u} = (f_1(x,y,t),\ f_2(x,y,t))$
- cost potential function: $\phi(x,y,t)$ gives the minimum instantaneous travel cost to reach the destination.
- divergence: $\nabla\cdot\mathbf{f} = \frac{\partial}{\partial x}f_1 + \frac{\partial}{\partial y}f_2$
- gradient: $\nabla\phi = (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y})$.
  Note: the gradient vector is always orthogonal, or normal, to the surface at a point.
- norm: $\|\mathbf{f}\| =\sqrt{f_1^2 + f_2^2}$

The conservation of travelers yields

$$\rho_t + \nabla\cdot\mathbf{f} = 0. \qquad \text{(2D Conservation Law)} \qquad (6.20)$$

Hughes makes three hypotheses in order to model the nature of the pedestrian flow:

1. the speed of pedestrians at each point is an isotropic function of the density at that point and time

   $$U(x,y,t) = \|\mathbf{u}\| = V(\rho), \qquad \text{(isotropic case)} \qquad (6.21)$$

   where $V(\rho)$ represent the speed-density relationships and plays a role of the MFD.
2. pedestrian motion is in the direction with maximum potential reduction, i.e. in the direction *perpendicular* to the potential or, opposite to the gra dient. Thus, the velocity vector is given by

   $$\mathbf{u} = -\frac{\nabla\phi}{\|\nabla\phi\|}\,V(\rho), \qquad \text{(speed vector parallel to streamlines)} \qquad (6.22)$$

   where $\nabla\phi/\|\nabla\phi\|$ is the unit gradient vector of the potential function.
3. Discomfort function, $g(\rho)$ to avoid higher densities, which satisfies

   $$\frac{1}{\|\nabla\phi\|} = gV, \qquad \text{(discomfort function } g\text{)} \qquad (6.23)$$

   where

   $$g \geqslant 1, \quad \frac{\partial g}{\partial\rho} \geqslant 0.$$

With all, Hughes model' can be expressed as

$$\begin{cases} \rho_t - \nabla\cdot\left(\rho\, g V^2 \nabla\phi\right) = 0, & \text{(Hughes' model)} \quad (6.24a)\\ gV = 1/||\nabla\phi||, & (6.24b) \end{cases}$$

which is not a hyperbolic system, as customary in the 1D traffic flow literature.

6.6.1 Reactive Dynamic User Equilibrium

In the Reactive Dynamic User Equilibrium (RDUE) problem, travelers choose the route that minimizes the instantaneous travel cost and change their choice en route as a result.

Huang et al.[0] demonstrated that Hughes' assumption 2 satisfies the RDUE principle.

Let $c(x,y,t)$ be the local cost per unit distance of movement:

$$c = \|\nabla\phi\|, \tag{6.25}$$

which is an Eikonal equation.

Since $\mathbf{f}/\|\mathbf{f}\|$ is the unit vector of the direction of movement, to minimize the instantaneous walking cost of a traveler, the route choice strategy should satisfy the following equation

$$c\frac{\mathbf{f}}{\|\mathbf{f}\|} + \nabla\phi = 0, \qquad \text{(reactive route-choice strategy)} \tag{6.26}$$

which means that the flux vector, and as a result the velocity vector, are parallel to (indicated by //) the gradient of the cost potential and in the opposite direction

$$\mathbf{u} \;//\; \mathbf{f} \;//\; -\nabla\phi. \qquad \text{(optimum movement direction)} \tag{6.27}$$

They propose a general modeling structure consisting a set of partial differential equations as

$$\begin{cases} \rho_t + \nabla\cdot\mathbf{f} = 0, & \forall(x,y)\in\Omega, \quad (6.28a) \\ \|\mathbf{f}\| = \rho U, & \forall(x,y)\in\Omega, \quad (6.28b) \\ c\dfrac{\mathbf{f}}{\|\mathbf{f}\|} + \nabla\phi = 0, & \forall(x,y)\in\Omega, \quad (6.28c) \end{cases}$$

subject to appropriate initial and boundary conditions, typically:

$$\begin{cases} \rho(x,y,0) = \rho_0(x,y), & \forall(x,y)\in\Omega, \quad (6.29a) \\ \mathbf{f}\cdot\hat{\mathbf{n}}(x,y) = q(x,y,t), & \forall(x,y)\in\Gamma_o, \quad (6.29b) \\ \phi = 0, & \forall(x,y)\in\Gamma_d, \quad (6.29c) \end{cases}$$

where $\hat{\mathbf{n}}(x,y)$ is the unit normal vector toward Ω on Γ_o, $q(x,y,t)$ denotes the time-varying demand and $\rho_0(x,y)$ is the density of each point inside the modeling domain at the beginning of the modeling period.

6.7 Problems

Problem 6.1 — A CBD rush hour During the morning rush hour vehicles enter the CBD at 500 vehicles per minute. The free-flow travel time until reaching their destination somewhere inside the CBD is 20 min, the CBD has a jam accumulation of 10,000 vehicles, a free-flow speed of 40 km/hr and a maximum average flow of 200 vehicles per min.

At $t = 0$ the CBD is empty and there is an incident on the network that decreases its maximum average flow to 100 vehicles per min.

a) At what time will the system reach equilibrium, and what is the equilibrium accumulation it reaches?
b) What is the travel time at $t = 80$ min?

After reaching equilibrium, the incident is removed:

c) How long after the incident is removed does the system reach a new equilibrium and what is the equilibrium accumulation it reaches?
d) What is the travel time 40 min after the incident is removed ?